M340(921) Solutions—Practice Problems

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1. Consider a zero-sum game between Claude and Rachel in which the following matrix shows Claude's winnings:

$$G = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} -4 & 2 & 5 \\ 2 & -4 & -3 \\ 3 & -6 & -2 \\ -3 & 8 & 6 \end{pmatrix} \end{array}$$

(a) Explain why Rachel will never choose R_4. Use this fact to reduce the game to one involving a 3×3 matrix.

(b) Use similar thinking to reduce the game to one involving a 2×2 matrix.

(c) Solve (without computer assistance) the 2×2 matrix game found in (b). Then use your answer to reveal the optimal strategies for Rachel and Claude in the original 4×3 game.

Our game matrix is

$$A = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} -4 & 2 & 5 \\ 2 & -4 & -3 \\ 3 & -6 & -2 \\ -3 & 8 & 6 \end{pmatrix} \end{array}$$

(a) If Rachel chooses R_1 instead of R_4, she will have to pay less to Claude *no matter which strategy he might use.* This is reflected in the following element-by-element inequality between row vectors:

$$[\,-4 \quad 2 \quad 5\,] \leq [\,-3 \quad 8 \quad 6\,].$$

Any strategy for Rachel that puts positive probability on R_4 will be strictly improved if she shifts that probability onto R_3 instead. So Rachel can forget about ever choosing R_4. Claude can imagine Rachel's reasoning, and realize for himself that she will never use R_4. Thus both players might as well be looking at the 3×3 game matrix

$$A_1 = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} -4 & 2 & 5 \\ 2 & -4 & -3 \\ 3 & -6 & -2 \end{pmatrix} \end{array}$$

(b) Claude likes C_3 better than C_2 because, *no matter what Rachel might do,* Claude's payoff from C_3 is better than from C_2. This is captured in the following element-by-element inequality between column vectors:

$$\begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix} \leq \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix}.$$

So Claude will never play C_2, Rachel can predict his reasoning, and so both players will act as though it is not in consideration. Deleting C_2 produces a 3×2 game matrix. But then Rachel looks at the remaining three rows and observes the element-by-element inequality

$$[\,2 \quad -3\,] \leq [\,3 \quad -2\,].$$

She realizes that R_3 is always a less attractive strategy than R_2, so she deletes R_3. Claude can predict that this is the rational thing to do. Thus both players arrive at the matrix

$$\widetilde{A} = \begin{array}{c} \\ R_1 \\ R_2 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_3 \end{array} \\ \begin{pmatrix} -4 & 5 \\ 2 & -3 \end{pmatrix} \end{array}.$$

Now there is no obvious advantage to either player, so hard work will be required to finish.

(c) Claude's problem is captured by a certain LP. To make it easy to translate the results back to the full-sized problem, let's use the unusual notation $\mathbf{x} = (x_1, x_3)$ for Claude's reduced choice vector.

$$\begin{aligned} \text{maximize} \quad & f = v \\ \text{subject to} \quad & v + 4x_1 - 5x_3 \le 0 \\ & v - 2x_1 + 3x_3 \le 0 \\ & x_1 + \ \ x_3 = 1 \\ & v \text{ free, } \mathbf{x} \ge \mathbf{0} \text{ in } \mathbb{R}^2 \end{aligned}$$

Substituting $x_3 = 1 - x_1$ leads to

$$\begin{aligned} \text{maximize} \quad & f = v \\ \text{subject to} \quad & v + 9x_1 \le \ \ 5 \\ & v - 5x_1 \le -3 \\ & x_1 \le \ \ 1 \\ & v \text{ free, } x_1 \ge 0 \end{aligned}$$

A dictionary for this problem, with slack variables named w_1, w_2, and (!) x_3, is

$$\begin{array}{rcl} f & = & v \\ \hline w_1 & = & 5 - 9x_1 - v \\ w_2 & = & -3 + 5x_1 - v \\ x_3 & = & 1 - \ \ x_1 \end{array}$$

We pivot v into the basis and w_2 out to gain feasibility. Then we decline to write v in the dictionary, because $f = v$ will track it for us.

$$\begin{array}{rcl} f & = & -3 + \ 5x_1 - w_2 \\ \hline x_3 & = & 1 - \ \ \ x_1 \\ w_1 & = & 8 - 14x_1 + w_2 \end{array}$$

In the next pivot, x_1 enters the basis and w_1 leaves, producing

$$\begin{array}{rcl} f & = & -1/7 - (5/14)w_1 - (9/14)w_2 \\ \hline x_1 & = & 4/7 - (1/14)w_1 + (1/14)w_2 \\ x_3 & = & 3/7 + (1/14)w_1 - (1/14)w_2 \end{array}$$

This is optimal. The value of the game is $-1/7$ (Claude loses on average). Claude's optimal strategy in the full original game is

$$\mathbf{x}^* = \left(\frac{4}{7}, 0, \frac{3}{7}\right).$$

We can find Rachel's optimal strategy in the original game by reading the dual coefficients in the optimal dictionary above, and positioning them appropriately:

$$\mathbf{y}^* = \left(\frac{5}{14}, \frac{9}{14}, 0, 0\right).$$

Check: The strategy vectors $\mathbf{x}^*$ and $\mathbf{y}^*$ satisfy $\text{minElement}(G\mathbf{x}^*) = \text{maxElement}((\mathbf{y}^*)^T G)$, as the following calculations confirm.

$$A\mathbf{x}^* = \frac{1}{7}\begin{bmatrix} -4 & 2 & 5 \\ 2 & -4 & -3 \\ 3 & -6 & -2 \\ -3 & 8 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -1 \\ -1 \\ 6 \\ 4 \end{bmatrix} \implies \text{minElement}(A\mathbf{x}^*) = -\frac{1}{7},$$

$$(\mathbf{y}^*)^T A = \frac{1}{14}\begin{bmatrix} 5 & 9 & 0 & 0 \end{bmatrix}\begin{bmatrix} -4 & 2 & 5 \\ 2 & -4 & -3 \\ 3 & -6 & -2 \\ -3 & 8 & 6 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -2 & -13 & -2 \end{bmatrix} \implies \text{maxElement}((\mathbf{y}^*)^T A) = -\frac{1}{7}.$$

Therefore these vectors are in Nash equilibrium, and the value of the game is also recovered as $-\frac{1}{7}$.

2. Given a fixed positive integer N, Rory and Cleo play a simple game. Rory secretly chooses an integer from the set $\{1, 2, \ldots, N\}$, and Cleo guesses what he has chosen. If Cleo's guess is correct, she wins 5 Galactic Currency Units; Rory pays her those. If her guess is not correct, no currency changes hands.

Find the optimal strategies for both players, and the expected payout to Cleo.

Suggestion: An efficient approach is to make smart conjectures about the desired quantities, and then to confirm the correctness of your proposals by applying well-known theorems.

Cleo should guess all values with equal probability. Rory should choose all values with equal probability. Cleo will guess correctly with probability $1/N$, so her expected payout is $5/N$.

To confirm this, note that the game matrix is $A = 5I$ and the strategies just described are the vectors

$$\mathbf{x}^* = \frac{1}{N}\mathbf{1}, \qquad \mathbf{y}^* = \frac{1}{N}\mathbf{1}.$$

One has

$$\begin{aligned} (\mathbf{x}^*)^T A = &\ \frac{1}{N}\mathbf{1}^T(5I) = \frac{5}{N}\mathbf{1}^T, \quad \text{so} \quad \text{MinElement}((\mathbf{x}^*)^T A) = \frac{5}{N}, \\ A\mathbf{y}^* = (5I)\left(\frac{1}{N}\mathbf{1}\right) = &\ \frac{5}{N}\mathbf{1}, \quad \text{so} \quad \text{MaxElement}(A\mathbf{y}^*) = \frac{5}{N}. \end{aligned}$$

Thus $\mathbf{x}^*$ and $\mathbf{y}^*$ are probability vectors satisfying

$$\text{MinElement}((\mathbf{x}^*)^T A) = \text{MaxElement}(A\mathbf{y}^*) = \frac{5}{N}.$$

By Weak Duality, this is an equilibrium pair with the indicated value.

3. Rowena plays the rows and Callum plays the columns in a standard zero-sum matrix game in which the rewards to Callum are displayed in the following matrix:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 2 & 1 \\ -1 & 4 \\ -2 & 5 \end{bmatrix}.$$

(a) Write a short, clear definition of "Nash equilibrium" applicable to zero-sum games. Use only words: no mathematical symbols or variables are allowed.

(b) Consider the mixed strategies $\widetilde{\mathbf{x}} = \left(\frac{1}{4}, \frac{3}{4}\right)$ for Callum and $\mathbf{y}^* = \left(0, 0, \frac{5}{6}, \frac{1}{6}, 0\right)$ for Rowena. Are these strategies in Nash equilibrium? Explain, making reference to your definition in part (a).

(c) Find all strategies $\mathbf{x}$ for Callum (if any) that can participate in a Nash equilibrium with Rowena's choice of $\mathbf{y}^*$ from (b).

(a) Two strategies are in Nash equilibrium when there is no incentive for either player to make a unilateral change of strategy. Alternatively, neither strategy has an obvious exploitable flaw.

(b) Callum's proposed strategy $\widetilde{\mathbf{x}} = \left(\frac{1}{4}, \frac{3}{4}\right)$ presents to Rowena the average column vector

$$A\widetilde{\mathbf{x}} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 2 & 1 \\ -1 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \left(\frac{1}{4}\right) = \frac{1}{4} \begin{bmatrix} 9 \\ 7 \\ 5 \\ 11 \\ 13 \end{bmatrix} \implies \mathrm{MinElement}(A\widetilde{\mathbf{x}}) = \frac{5}{4}.$$

Meanwhile, Rowena's suggested strategy $\mathbf{y}^* = \left(0, 0, \frac{5}{6}, \frac{1}{6}, 0\right)$ presents to Callum the average row vector

$$(\mathbf{y}^*)^T A = \left(\frac{1}{6}\right) \begin{bmatrix} 0 & 0 & 5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 2 & 1 \\ -1 & 4 \\ -2 & 5 \end{bmatrix} = \left(\frac{1}{6}\right) \begin{bmatrix} 9 & 9 \end{bmatrix} \implies \mathrm{MaxElement}((\mathbf{y}^*)^T A) = \frac{3}{2}.$$

Since $\frac{5}{4} \neq \frac{3}{2}$, this pair is not in Nash equilibrium. Rowena's strategy vector puts some positive probability on element 4 of $A\widetilde{\mathbf{x}}$. She could do better against $\widetilde{\mathbf{x}}$ by playing $(0, 0, 1, 0, 0)$ instead. This is the sort of "unilateral change in strategy" that the definition forbids.

(c) Given Rowena's strategy $\mathbf{y}^*$, the only chance for Callum to play a strategy $\mathbf{x}^*$ that is in equilibrium with Rowena's choice is for $\mathbf{x}^* = (x_1, x_2)$ to obey

$$\mathrm{MinElement}(A\mathbf{x}^*) = \mathrm{MaxElement}((\mathbf{y}^*)^T A) = \frac{3}{2} \qquad \text{and} \qquad A\mathbf{x}^* = \begin{bmatrix} ? \\ ? \\ 3/2 \\ 3/2 \\ ? \end{bmatrix}.$$

Here the smallest elements in $A\mathbf{x}^*$ are located in the components where $\mathbf{y}^*$ assigns positive probability. This leads to the 2×2 system

$$\left\{ \begin{aligned} 2x_1 + \ x_2 &= 3/2 \\ -x_1 + 4x_2 &= 3/2 \end{aligned} \right\} \iff \left\{ \begin{aligned} x_1 &= 1/2 \\ x_2 &= 1/2 \end{aligned} \right\}$$

With these choices, $\mathbf{x}^* = (\frac{1}{2}, \frac{1}{2})$ satisfies

$$A\mathbf{x}^* = \frac{1}{2}\begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 2 & 1 \\ -1 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \implies \mathrm{MinElement}(A\mathbf{x}^*) = \frac{3}{2}.$$

This meets all the requirements articulated above, so the pair $\mathbf{x}^*$, $\mathbf{y}^*$ is in Nash equilibrium.

4. Claude and Rachel play a zero-sum matrix game with the usual rules: Claude uses a vector $\mathbf{x} \in \mathbb{P}_3$ to play the columns, Rachel uses a vector $\mathbf{y} \in \mathbb{P}_2$ to play the rows, and Rachel pays Claude $\mathbf{y}^T A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}.$$

(a) Find the optimal strategies for both players.

(b) Is the game fair? If so, prove it; if not, say which player the game favours.

(a) Claude's problem is to maximize MinElement($A\mathbf{x}$) over all probability vectors $\mathbf{x} \in \mathbb{P}(3)$. He can express this as a linear program:

$$\begin{aligned} \text{maximize } & f = v \\ \text{subject to } \quad & v - 2x_1 + x_2 \phantom{{}-x_3} \le 0 \\ & v + 2x_1 - 3x_2 - x_3 \le 0 \\ & \phantom{v+{}} x_1 + x_2 + x_3 = 1 \\ & v \in \mathbb{R} \text{ free; } \mathbf{x} \ge \mathbf{0} \text{ in } \mathbb{R}^3 \end{aligned}$$

Substitute $x_3 = 1 - x_1 - x_2$:

$$\begin{aligned} \text{maximize } & f = v \\ \text{subject to } \quad & v - 2x_1 + x_2 \le 0 \\ & v + 3x_1 - 2x_2 \le 1 \\ & \phantom{v+3{}} x_1 + x_2 \le 1 \\ & v \in \mathbb{R} \text{ free; } (x_1, x_2) \ge (0, 0) \end{aligned}$$

Use slacks w_1, w_2, and $w_3 \equiv x_3$ to build a dictionary:

$$\begin{array}{rcl} f & = & \phantom{1-{}} v \\ \hline w_1 & = & - v + 2x_1 - x_2 \\ w_2 & = & 1 - v - 3x_1 + 2x_2 \\ x_3 & = & 1 \phantom{{}-v} - x_1 - x_2 \end{array}$$

This is improvable because v can have either sign. So put v into the basis. For feasibility, pivot w_1 out, using

$$v = 2x_1 - x_2 - w_1.$$

Since $f = v$ always, there is no need to keep a second copy of the equation for v. Drop it now. The new dictionary is

$$\begin{array}{rcl} f & = & - w_1 + 2x_1 - x_2 \\ \hline x_3 & = & 1 \phantom{{}+w_1} - x_1 - x_2 \\ w_2 & = & 1 + w_1 - 5x_1 + 3x_2 \end{array}$$

Pivot x_1 with w_2 to arrive at

$$\begin{array}{rcl} f & = & (2/5) + (1/5)x_2 - (3/5)w_1 - (2/5)w_2 \\ \hline x_1 & = & (1/5) + (3/5)x_2 + (1/5)w_1 - (1/5)w_2 \\ x_3 & = & (4/5) - (8/5)x_2 - (1/5)w_1 + (1/5)w_2 \end{array}$$

Now x_2 must enter and x_3 must leave, via the pivot equation

$$x_2 = \frac{1}{2} - \frac{5}{8}x_3 - \frac{1}{8}w_1 + \frac{1}{8}w_2.$$

The objective function updates to become

$$f = \frac{1}{2} - \frac{1}{8}x_3 - \frac{5}{8}w_1 - \frac{3}{8}w_2.$$

The negative non-basic coefficients indicate an optimal dictionary, and we don't really need all the details. The optimal values will be

$$\mathbf{x}^* = \left(\frac{1}{2}, \frac{1}{2}, 0\right),$$

and the corresponding dual solutions will be $\mathbf{y}^* = \left(\frac{5}{8}, \frac{3}{8}\right)$. To check that this all works, notice the equality between the values

$$(\mathbf{y}^*)^T A = \frac{1}{8}\begin{bmatrix} 5 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{3}{8} \end{bmatrix} \implies \text{MaxElement}((\mathbf{y}^*)^T A) = \frac{1}{2}$$

$$A\mathbf{x}^* = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \implies \text{MaxElement}(A\mathbf{x}) = \frac{1}{2}$$

(b) The game is NOT fair. The maximum value for f, Claude's reward, is positive, so the game favours Claude.

5. The game of Morra, which goes back as far as Roman times, has two players. Each player shows either one or two fingers and simultaneously tells their opponent a guess at how many fingers the opponent will show. If both guesses are right or both are wrong, the game is a tie and no gold changes hands. But if one player guesses right and the other guesses wrong, the correct player wins. The winning strategy (show s, tell t) determines the prize: the loser pays the winner $s+t$ gold pieces.

(a) Use the notation $[s,t]$ to encode the strategy of showing s and telling t. Fill in the missing entries in this 4×4 game matrix showing the payoff to the column player:

$$\begin{array}{c c} & \begin{array}{cccc} [1,1] & [1,2] & [2,1] & [2,2] \end{array} \\ \begin{array}{c} [1,1] \\ [1,2] \\ [2,1] \\ [2,2] \end{array} & \left(\begin{array}{cccc} 0 & -2 & 3 & 0 \\ & & & \\ & & & \\ & & & \end{array}\right) \end{array}$$

(b) Find the best mixed strategy for this game by setting up and solving a suitable LP. (Computer assistance is allowed.)

(a) Here is Claude's payoff matrix A in 2-finger Morra:

$$\begin{array}{c c} & \begin{array}{cccc} [1,1] & [1,2] & [2,1] & [2,2] \end{array} \\ \begin{array}{c} [1,1] \\ [1,2] \\ [2,1] \\ [2,2] \end{array} & \left(\begin{array}{cccc} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{array}\right) \end{array}$$

(b) Written in full detail, here is Claude's LP:

$$\begin{array}{ll} \text{maximize} & f = v \\ \text{subject to} & v \qquad + 2x_2 - 3x_3 \qquad \le 0 \\ & v - 2x_1 \qquad\qquad + 3x_4 \le 0 \\ & v + 3x_1 \qquad\qquad - 4x_4 \le 0 \\ & v \qquad - 3x_2 + 4x_3 \qquad \le 0 \\ & \quad x_1 + x_2 + x_3 + x_4 = 1 \\ & v \text{ free; } \mathbf{x} \ge \mathbf{0} \text{ in } \mathbb{R}^4 \end{array}$$

Substituting $x_4 = 1 - x_1 - x_2 - x_3$ leads to a problem where the equality constraint is removed:

$$\begin{array}{ll} \text{maximize} & f = v \\ \text{subject to} & v \qquad + 2x_2 - 3x_3 \le 0 \\ & v - 5x_1 - 3x_2 - 3x_3 \le -3 \\ & v + 7x_1 + 4x_2 + 4x_3 \le 4 \\ & v \qquad - 3x_2 + 4x_3 \le 0 \\ & \quad x_1 + x_2 + x_3 \le 1 \\ & v \text{ free; } \mathbf{x} \ge \mathbf{0} \text{ in } \mathbb{R}^3 \end{array}$$

Introduce slack variables named w_1, w_2, w_3, w_4, x_4 to get the initial dictionary

$$\begin{array}{rcl} f &=& v \\ \hline w_1 &=& -v \qquad - 2x_2 + 3x_3 \\ w_2 &=& -3 - v + 5x_1 + 3x_2 + 3x_3 \\ w_3 &=& 4 - v - 7x_1 - 4x_2 - 4x_3 \\ w_4 &=& -v \qquad + 3x_2 - 4x_3 \\ x_4 &=& 1 \qquad - x_1 - x_2 - x_3 \end{array}$$

Now pivot v into the basis, using w_2 as the leaving variable to cure infeasibility. As discussed in class, the identity in the objective row makes it redundant to write the pivot equation anywhere else. Thus we arrive at

$$\begin{array}{rcl} f &=& -3 + 5x_1 + 3x_2 + 3x_3 - w_2 \\ \hline x_4 &=& 1 - x_1 - x_2 - x_3 \\ w_1 &=& 3 - 5x_1 - 5x_2 + w_2 \\ w_3 &=& 7 - 12x_1 - 7x_2 - 7x_3 + w_2 \\ w_4 &=& 3 - 5x_1 - 7x_3 + w_2 \end{array}$$

This is a form in which we can apply a simple standard online LP solver. It suffices to maximize the objective $z = f + 3$ subject to nonnegativity constraints on the basic variables written above. Here is some suitable input:

```
maximize z = 5 x1 + 3 x2 + 3 x3 - w2
subject to
                x1 +   x2 +   x3      <=  1
              5 x1 + 5 x2        - w2 <=  3
             12 x1 + 7 x2 + 7 x3 - w2 <=  7
              5 x1        + 7 x3 - w2 <=  3
```

The solver (in "Fraction" mode) comes back with this:

```
Optimal Solution:  z = 3; x1 = 0, x2 = 3/5, x3 = 2/5, w2 = 0
```

Therefore $f_{\max} = -3 + z_{\max} = 0$. (The game is fair.) An optimal strategy for the column player is

$$\mathbf{x}^* = (0, 3/5, 2/5, 0)\,.$$

Notice that the game matrix satisfies $A^T = -A$, so the game problem is self-dual. The optimal strategy for the row player is identical to the one for the column player: $\mathbf{y}^* = \mathbf{x}^* = (0, 3/5, 2/5, 0)$.

Remark. The online solver in fraction mode confirms what hand-calculation also indicates: it takes about three pivots to get from the dictionary above to the optimal dictionary, and this process leads through fractions having denominators like 12, 25, and 7.

6. Having learned about "2-finger Morra" in the previous question, it should be easy to infer the rules for "N-finger Morra." Consider the case where $N = 3$.

(a) Using the notation $[s, t]$ to encode the strategy of showing s and telling t, fill in the 9×9 game matrix showing the payoff to the column player.

$$
\begin{array}{c}
\\
{[1,1]} \\ {[1,2]} \\ {[1,3]} \\ {[2,1]} \\ {[2,2]} \\ {[2,3]} \\ {[3,1]} \\ {[3,2]} \\ {[3,3]}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
[1,1] & [1,2] & [1,3] & [2,1] & [2,2] & [2,3] & [3,1] & [3,2] & [3,3]
\end{array} \\
\begin{pmatrix}
0 & -2 & -2 & 3 & 0 & 0 & 4 & 0 & 0 \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & &
\end{pmatrix}
\end{array}
$$

(b) Show that playing each possible strategy with probability 1/9 is not optimal.

(c) If you were confronted with an opponent playing the strategy in (b), what strategy would you adopt? Why? Is your choice unique?

(d) Verify that the optimal strategy for both players is the probability vector

$$\mathbf{p} = (0, 0, 5/12, 0, 1/3, 0, 1/4, 0, 0)\,.$$

Note: Checking a given vector for optimality ought to be much easier than deriving a solution from first principles. This remark applies to part (b) as well as to part (d).

(a) Here are the zero-entries identifying cases where either both players guess right or both guess wrong. Notice the symmetric pattern.

$$
\begin{array}{c|ccccccccc}
 & [1,1] & [1,2] & [1,3] & [2,1] & [2,2] & [2,3] & [3,1] & [3,2] & [3,3] \\
\hline
{[1,1]} & 0 & & & & 0 & 0 & & 0 & 0 \\
{[1,2]} & & 0 & 0 & 0 & & & & 0 & 0 \\
{[1,3]} & & 0 & 0 & & 0 & 0 & 0 & & \\
{[2,1]} & & 0 & & 0 & & 0 & 0 & & 0 \\
{[2,2]} & 0 & & 0 & & 0 & & 0 & & 0 \\
{[2,3]} & 0 & & 0 & 0 & & 0 & & 0 & \\
{[3,1]} & & & 0 & 0 & 0 & & 0 & 0 & \\
{[3,2]} & 0 & 0 & & & & 0 & 0 & 0 & \\
{[3,3]} & 0 & 0 & & 0 & 0 & & & & 0
\end{array}
$$

Next we fill in the payoffs to Claude when he (playing the columns) wins:

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} [1,1] & [1,2] & [1,3] & [2,1] & [2,2] & [2,3] & [3,1] & [3,2] & [3,3] \end{array} \\
\begin{array}{c} [1,1] \\ [1,2] \\ [1,3] \\ [2,1] \\ [2,2] \\ [2,3] \\ [3,1] \\ [3,2] \\ [3,3] \end{array} &
\begin{pmatrix}
0 & & & 3 & 0 & 0 & 4 & 0 & 0 \\
2 & 0 & 0 & 0 & & & 4 & 0 & 0 \\
2 & 0 & 0 & 3 & 0 & 0 & 0 & & \\
 & 0 & & 0 & 4 & 0 & 0 & 5 & 0 \\
0 & 3 & 0 & & 0 & & 0 & 5 & 0 \\
0 & 3 & 0 & 0 & 4 & 0 & & 0 & \\
 & & 0 & 0 & 0 & 5 & 0 & 0 & 6 \\
0 & 0 & 4 & & & 0 & 0 & 0 & 6 \\
0 & 0 & 4 & 0 & 0 & 5 & & & 0
\end{pmatrix}
\end{array}
$$

The game is symmetric, so $A^T = -A$. This gives a consistency check as we fill in the entries where Rachel wins (Claude loses) to complete the matrix:

$$
A = \begin{array}{cc}
 & \begin{array}{ccccccccc} [1,1] & [1,2] & [1,3] & [2,1] & [2,2] & [2,3] & [3,1] & [3,2] & [3,3] \end{array} \\
\begin{array}{c} [1,1] \\ [1,2] \\ [1,3] \\ [2,1] \\ [2,2] \\ [2,3] \\ [3,1] \\ [3,2] \\ [3,3] \end{array} &
\begin{pmatrix}
0 & -2 & -2 & 3 & 0 & 0 & 4 & 0 & 0 \\
2 & 0 & 0 & 0 & -3 & -3 & 4 & 0 & 0 \\
2 & 0 & 0 & 3 & 0 & 0 & 0 & -4 & -4 \\
-3 & 0 & -3 & 0 & 4 & 0 & 0 & 5 & 0 \\
0 & 3 & 0 & -4 & 0 & -4 & 0 & 5 & 0 \\
0 & 3 & 0 & 0 & 4 & 0 & -5 & 0 & -5 \\
-4 & -4 & 0 & 0 & 0 & 5 & 0 & 0 & 6 \\
0 & 0 & 4 & -5 & -5 & 0 & 0 & 0 & 6 \\
0 & 0 & 4 & 0 & 0 & 5 & -6 & -6 & 0
\end{pmatrix}
\end{array}
$$

(b) For the uniform strategy $(1/9)\mathbf{1}$, Claude's objective is

$$
f\left(\frac{1}{9}\mathbf{1}\right) = \frac{1}{9}\text{MinElement}\left(A\mathbf{1}\right) = \frac{1}{9}\text{MinElement}\begin{bmatrix} 3 \\ 0 \\ -3 \\ 3 \\ 0 \\ -3 \\ 3 \\ 0 \\ -3 \end{bmatrix} = -\frac{1}{3}.
$$

But this is a symmetric game, so its value must be 0. Thus there must exists some strategy $\mathbf{x}^*$ for Claude for which $f(\mathbf{x}^*) = 0$. Certainly the uniform strategy is not optimal.

(c) If Claude plays the uniform strategy $(1/9)\mathbf{1}$, Rachel should counter with a strategy of the form $\mathbf{y} = (0, 0, y_3, 0, 0, y_6, 0, 0, y_9)$ where $(y_3, y_6, y_9) \in \mathbb{P}_3$. Any such strategy will produce an average payout per round of

$$\mathbf{y}^T A \left(\frac{1}{9}\mathbf{1}\right) = -\frac{1}{3}(y_3 + y_6 + y_9) = -\frac{1}{3}.$$

This penalizes Claude (favours Rachel) in the strongest possible way.

(d) The arithmetic is simpler if we write the candidate strategy as

$$\mathbf{p} = \frac{1}{12}\left(0, 0, 5, 0, 4, 0, 3, 0, 0\right).$$

Then $\mathbf{x}^* = \mathbf{p}$ leads to $A\mathbf{x}^* = \dfrac{1}{12}\left(2, 0, 0, 1, 0, 1, 0, 0, 2\right)$, so

$$\mathrm{MinElement}\left(A\mathbf{x}^*\right) = 0. \tag{$*$}$$

And $\mathbf{y}^* = \mathbf{p}$ leads to $(\mathbf{y}^*)^T A = \dfrac{1}{12}\left(-2, 0, 0, -1, 0, -1, 0, 0, -2\right)$, so

$$\mathrm{MaxElement}\left((\mathbf{y}^*)^T A\right) = 0. \tag{$**$}$$

Equality between the scalar values in $(*)$ and $(**)$ confirms the simultaneous optimality of $\mathbf{x}^*$ and $\mathbf{y}^*$ for the two players.

7. Different choices for k make for different outcomes in the zero-sum matrix game defined by

$$A(k) = \begin{bmatrix} 2 & 3 \\ k & 1 \\ 3 & 2 \end{bmatrix}.$$

(Recall: The row player, Rory, plays $\mathbf{y} \in \mathbb{P}(3)$; the column player, Cleo, plays $\mathbf{x} \in \mathbb{P}(2)$; the reward to Cleo is $\mathbf{y}^T A(k)\mathbf{x}$.)

(a) Find an equilibrium pair of strategies, and Cleo's reward, when $k = 2$.

(b) Find an equilibrium pair of strategies, and Cleo's reward, when $k = 3$.

(c) Find the largest interval of k-values around $k = 3$ with this property: Rory's optimal strategy has the same support as it does when $k = 3$. Find the equilibrium strategies and the game's value as a function of k in this interval.

(a) The entries in each row reveal Rory's losses if he plays that row. When $k = 2$, row 2 provides a smaller loss in every scenario than both columns 1 and 3. So Rory can use the pure strategy $\mathbf{y}^* = (0, 1, 0)$. Cleo, the column player, will respond with the pure strategy $\mathbf{x}^* = (1, 0)$, and earn $k = 2$. In fact, this reasoning applies whenever $1 \le k \le 2$. To prove this, apply weak duality:

$$(\mathbf{y}^*)^T A(k) = [\,k \quad 1\,] \implies \mathrm{MaxElement}\left((\mathbf{y}^*)^T A(k)\right) = k;$$

$$A(k)\mathbf{x}^* = \begin{bmatrix} 2 \\ k \\ 3 \end{bmatrix} \implies \mathrm{MinElement}\left(A(k)\mathbf{x}^*\right) = k.$$

Since $\mathrm{MinElement}\left((\mathbf{x}^*)^T G(k)\right) = \mathrm{MaxElement}\left(G(k)\mathbf{y}^*\right) = k$, we have an equilibrium pair with value k. This works whenever $1 \le k \le 2$. (When $k = 2$, there are many other equilibrium pairs: $\mathbf{x}^* = (1, 0)$ works with $\mathbf{y}^* = (q, 1-q, 0)$ for each q obeying $0 \le q \le \frac{1}{2}$.)

(b) When $k = 3$, Rory will continue to prefer Row 2 over Row 3, so the game can be reduced to

$$\widetilde{A}(3) = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}.$$

Grinding calculation reveals that both players have $\left(\dfrac{2}{3}, \dfrac{1}{3}\right)$ as an optimal strategy. Back in the original formulation, the equilibrium pair is

$$\mathbf{x}^* = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}, \qquad \mathbf{y}^* = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \end{bmatrix}.$$

This can be checked with complementarity, and those operations are implicit in the solution to part (c) below.

(c) [Sorry, out of time for typing.]