Math 340(921) Practice Problems on Matrix Games

(not to hand in)

1. Consider a zero-sum game between Claude and Rachel in which the following matrix shows Claude's winnings:

$$G = \begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} -4 & 2 & 5 \\ 2 & -4 & -3 \\ 3 & -6 & -2 \\ -3 & 8 & 6 \end{pmatrix} \end{array}$$

(a) Explain why Rachel will never choose R_4. Use this fact to reduce the game to one involving a 3×3 matrix.

(b) Use similar thinking to reduce the game to one involving a 2×2 matrix.

(c) Solve (without computer assistance) the 2×2 matrix game found in (b). Then use your answer to reveal the optimal strategies for Rachel and Claude in the original 4×3 game.

2. Given a fixed positive integer N, Rory and Cleo play a simple game. Rory secretly chooses an integer from the set $\{1, 2, \ldots, N\}$, and Cleo guesses what he has chosen. If Cleo's guess is correct, she wins 5 Galactic Currency Units; Rory pays her those. If her guess is not correct, no currency changes hands.

Find the optimal strategies for both players, and the expected payout to Cleo.

Suggestion: An efficient approach is to make smart conjectures about the desired quantities, and then to confirm the correctness of your proposals by applying well-known theorems.

3. Rowena plays the rows and Callum plays the columns in a standard zero-sum matrix game in which the rewards to Callum are displayed in the following matrix:

$$G = \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 2 & 1 \\ -1 & 4 \\ -2 & 5 \end{bmatrix}.$$

(a) Write a short, clear definition of "Nash equilibrium" applicable to zero-sum games. Use only words: no mathematical symbols or variables are allowed.

(b) Consider the mixed strategies $\widetilde{\mathbf{x}} = \left(\frac{1}{4}, \frac{3}{4}\right)$ for Callum and $\mathbf{y}^* = \left(0, 0, \frac{5}{6}, \frac{1}{6}, 0\right)$ for Rowena. Are these strategies in Nash equilibrium? Explain, making reference to your definition in part (a).

(c) Find all strategies $\mathbf{x}$ for Callum (if any) that can participate in a Nash equilibrium with Rowena's choice of $\mathbf{y}^*$ from (b).

4. Claude and Rachel play a zero-sum matrix game with the usual rules: Claude uses a vector $\mathbf{x} \in \mathbb{P}_3$ to play the columns, Rachel uses a vector $\mathbf{y} \in \mathbb{P}_2$ to play the rows, and Rachel pays Claude $\mathbf{y}^T G \mathbf{x}$, where

$$G = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}.$$

(a) Find the optimal strategies for both players.

(b) Is the game fair? If so, prove it; if not, say which player the game favours.

5. The game of Morra, which goes back as far as Roman times, has two players. Each player shows either one or two fingers and simultaneously tells their opponent a guess at how many fingers the opponent will show.

More . . .

If both guesses are right or both are wrong, the game is a tie and no gold changes hands. But if one player guesses right and the other guesses wrong, the correct player wins. The winning strategy (show s, tell t) determines the prize: the loser pays the winner $s+t$ gold pieces.

(a) Use the notation $[s,t]$ to encode the strategy of showing s and telling t. Fill in the missing entries in this 4×4 game matrix showing the payoff to the column player:

$$\begin{array}{c} \\ [1,1] \\ [1,2] \\ [2,1] \\ [2,2] \end{array} \begin{array}{c} \begin{array}{cccc} [1,1] & [1,2] & [2,1] & [2,2] \end{array} \\ \begin{pmatrix} 0 & -2 & 3 & 0 \\ & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

(b) Find the best mixed strategy for this game by setting up and solving a suitable LP. (Computer assistance is allowed.)

6. Having learned about "2-finger Morra" in the previous question, it should be easy to infer the rules for "N-finger Morra." Consider the case where $N=3$.

(a) Using the notation $[s,t]$ to encode the strategy of showing s and telling t, fill in the 9×9 game matrix showing the payoff to the column player.

$$\begin{array}{c} \\ [1,1] \\ [1,2] \\ [1,3] \\ [2,1] \\ [2,2] \\ [2,3] \\ [3,1] \\ [3,2] \\ [3,3] \end{array} \begin{array}{c} \begin{array}{ccccccccc} [1,1] & [1,2] & [1,3] & [2,1] & [2,2] & [2,3] & [3,1] & [3,2] & [3,3] \end{array} \\ \begin{pmatrix} 0 & -2 & -2 & 3 & 0 & 0 & 4 & 0 & 0 \\ &&&&&&&& \\ &&&&&&&& \\ &&&&&&&& \\ &&&&&&&& \\ &&&&&&&& \\ &&&&&&&& \\ &&&&&&&& \\ &&&&&&&& \end{pmatrix} \end{array}$$

(b) Show that playing each possible strategy with probability 1/9 is not optimal.

(c) If you were confronted with an opponent playing the strategy in (b), what strategy would you adopt? Why? Is your choice unique?

(d) Verify that the optimal strategy for both players is the probability vector

$$\mathbf{p} = (0, 0, 5/12, 0, 1/3, 0, 1/4, 0, 0)\,.$$

Note: Checking a given vector for optimality ought to be much easier than deriving a solution from first principles. This remark applies to part (b) as well as to part (d).

7. Different choices for k make for different outcomes in the zero-sum matrix game defined by

$$A(k) = \begin{bmatrix} 2 & 3 \\ k & 1 \\ 3 & 2 \end{bmatrix}.$$

(Recall: The row player, Rory, plays $\mathbf{y} \in \mathbb{P}(3)$; the column player, Cleo, plays $\mathbf{x} \in \mathbb{P}(2)$; the reward to Cleo is $\mathbf{y}^T A(k)\mathbf{x}$.)

(a) Find an equilibrium pair of strategies, and Cleo's reward, when $k=2$.

(b) Find an equilibrium pair of strategies, and Cleo's reward, when $k=3$.

(c) Find the largest interval of k-values around $k=3$ with this property: the Rory's optimal strategy has the same support as it does when $k=3$. Find the equilibrium strategies and the game's value as a function of k in this interval.