M340(921) Solutions—Problem Set 6

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1. (a) If each pig is fed y_1 kilograms of corn, y_2 kilos of tankage, and y_3 kilos of alfalfa, the cost per pig is

$$g = 35y_1 + 30y_2 + 25y_3.$$

The nutritional minimums on carbohydrates, protein, and vitamins give, respectively,

$$\begin{aligned} 90y_1 + 20y_2 + 40y_3 &\geq 200 \\ 30y_1 + 80y_2 + 60y_3 &\geq 180 \\ 10y_1 + 20y_2 + 50y_3 &\geq 150 \end{aligned}$$

Of course, the feeding amounts cannot be negative. So the farmer's problem is

$$\begin{aligned} \text{minimize } g = 35y_1 + 30y_2 + 25y_3 & \\ \text{subject to} \quad 90y_1 + 20y_2 + 40y_3 &\geq 200 \\ 30y_1 + 80y_2 + 60y_3 &\geq 180 \\ 10y_1 + 20y_2 + 50y_3 &\geq 150 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

(b) A primal problem that would give the diet problem above as its dual is

$$\begin{aligned} \text{maximize } f = 200x_1 + 180x_2 + 150x_3 & \\ \text{subject to} \quad 90x_1 + \ 30x_2 + \ 10x_3 &\leq 35 \\ 20x_1 + \ 80x_2 + \ 20x_3 &\leq 30 \\ 40x_1 + \ 60x_2 + \ 50x_3 &\leq 25 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(c) Giving the online simplex solver the input

```
maximize  f = 200 x1 + 180 x2 + 150 x3
 90 x1 + 30 x2 + 10 x3 <= 35
 20 x1 + 80 x2 + 20 x3 <= 30
 40 x1 + 60 x2 + 50 x3 <= 25
```

leads to this response:

```
Optimal Solution:  f = 104.268; x1 = 0.365854, x2 = 0, x3 = 0.207317
```

The optimal tableau is also of interest:

x1	x2	x3	s1	s2	s3	f	
1	0.219512	0	0.0121951	0	-0.00243902	0	0.365854
0	55.122	0	-0.0487805	1	-0.390244	0	18.5366
0	1.02439	1	-0.0097561	0	0.0219512	0	0.207317
0	17.561	0	0.97561	0	2.80488	1	104.268

The bottom row of the tableau corresponds to the objective equation

$$f = 104.3 - 17.56x_2 - 0.9756s_1 - 2.805s_3.$$

(The online tool names the slack vector s.) The slack coefficients in the optimal objective row reveal the dual minimizer, so the solution to the diet problem in (a) is

$$y^* = (0.9756, 0, 2.805).$$

That is, each pig should get 0.9756 Kg of corn, 2.805 Kg of alfalfa, and no tankage at all. This will cost the farmer $g_{\min} = f_{\max} = 104.27$ cents per pig (each day).

Of course one can enter the problem from (a) directly:

```
minimize e = 35 y1 + 30 y2 + 25 y3
90 y1 + 20 y2 + 40 y3 >= 200
30 y1 + 80 y2 + 60 y3 >= 180
10 y1 + 20 y2 + 50 y3 >= 150
```

The response is consistent with our findings above.:

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Optimal Solution:  e = 104.268; y1 = 0.97561, y2 = 0, y3 = 2.80488
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The objective row in the optimal tableau (not shown here) includes the slack-variable coefficients that we saw earlier in the components of $x^* = (0.3659, 0, 0.2073)$.

(d) In any LP of standard symmetric form, the optimal dual values reveal the rate of change of the optimum objective value with respect to small variations in the constraint vector $\mathbf{b}$. To put the diet problem into standard symmetric form we would need to apply sign changes to *both* the constraints and the objective. These two changes cancel, so the optimal dual values for problem (a) reveal the rate of change of the optimal feeding cost with respect to the pigs' nutritional requirements. If the nutritional guarantees for the pigs were changed to

$$\begin{array}{ll} \text{Carbohydrates:} & 90y_1 + 20y_2 + 40y_3 \geq 200 + \alpha \\ \text{Protein:} & 30y_1 + 80y_2 + 60y_3 \geq 180 + \beta \\ \text{Vitamins:} & 10y_1 + 20y_2 + 50y_3 \geq 150 + \gamma \end{array}$$

then the minimum daily cost of feeding each one, in cents, would be

$$Z = 104.27 + 0.366\alpha + 0.207\gamma.$$

This relation is exact when the perturbation vector (α, β, γ) has small components. Promising the pigs more protein adds nothing to the cost because the current feeding strategy already gives them about 198 units—well over the 180 they are entitled to currently.

2. (a) This is a case for complementary slackness. The given problem has dual

$$\begin{array}{lrcl} \text{minimize } f = b_1y_1 + b_2y_2 + b_3y_3 & & & \\ \text{subject to} & 3y_1 + y_2 + 4y_3 & \geq & c_1 \\ & 2y_1 + y_2 + 3y_3 & \geq & c_2 \\ & y_1 + y_2 + 3y_3 & \geq & c_3 \\ & 2y_1 + y_2 + 4y_3 & \geq & c_4 \\ & y_1, y_2, y_3 \geq 0 & & \end{array}$$

(i) Primal Feasibility. Nonnegativity of $\mathbf{x}^* = (0, 2, 3, 0)$ and $\mathbf{w}^* = (0, 6, 0)$ are evident, and the prescribed values leave no flexibility in the choice of $\mathbf{b}$:

$$\begin{aligned} 0 = w_1^* &= b_1 - 3x_1^* - 2x_2^* - x_3^* - 2x_4^* = b_1 - 7 \implies b_1 = 7 \\ 6 = w_2^* &= b_2 - x_1^* - x_2^* - x_3^* - x_4^* = b_2 - 5 \implies b_2 = 11 \\ 0 = w_3^* &= b_3 - 4x_1^* - 3x_2^* - 3x_3^* - 4x_4^* = b_3 - 15 \implies b_3 = 15 \end{aligned}$$

(ii) Complementary Slackness, I. Since $w_2^* > 0$, we must have $y_2^* = 0$. Remember this.

(iii) Complementary Slackness, II. Positive primal decision variables require zero dual slacks. Thus

$$\begin{aligned} x_2^* > 0 &\implies 0 = z_2^* = 2y_1^* + y_2^* + 3y_3^* - c_2 = 2y_1^* + 3y_3^* - c_2 \\ x_3^* > 0 &\implies 0 = z_3^* = y_1^* + y_2^* + 3y_3^* - c_3 = y_1^* + 3y_3^* - c_3 \end{aligned}$$

This gives a 2×2 system of linear equations to solve for y_1^* and y_3^*:

$$\left.\begin{aligned} 2y_1^* + 3y_3^* &= c_2 \\ y_1^* + 3y_3^* &= c_3 \end{aligned}\right] \quad \iff \quad \left[\begin{aligned} y_1^* &= \quad c_2 - c_3 \\ y_3^* &= -\frac{1}{3}c_2 + \frac{2}{3}c_3 \end{aligned}\right.$$

(iv) Dual Feasibility. The vector $\mathbf{y}^* = (y_1^*, 0, y_3^*)$ will obey $\mathbf{y}^* \geq \mathbf{0}$ in $\mathbb{R}^3$ if and only if $y_1^* \geq 0$ and $y_3^* \geq 0$, i.e.,

$$c_2 \geq c_3 \qquad \text{and} \qquad c_2 \leq 2c_3.$$

Feasibility also requires $\mathbf{z}^* \geq 0$ in $\mathbb{R}^4$. Two components of this vector inequality have been arranged in part (iii); the others require

$$\begin{aligned} 0 \leq z_1^* &= 3y_1^* + 4y_3^* - c_1 = \frac{5}{3}c_2 - \frac{1}{3}c_3 - c_1 \\ 0 \leq z_4^* &= 2y_1^* + 4y_3^* - c_4 = \frac{2}{3}c_2 + \frac{2}{3}c_3 - c_4 \end{aligned}$$

Combining these two inequalities with the ones given above leads to the system

$$c_3 \leq c_2 \leq 2c_3, \qquad 5c_2 - c_3 \geq 3c_1, \qquad 2c_2 + 2c_3 \geq 3c_4. \tag{$*$}$$

The four inequalities in $(*)$ hold if and only if the full complementary slackness recipe works out, so they are logically equivalent to the statement "$\mathbf{x}^*$ achieves the maximum."

Notes (not required): **1.** Given some vector $\mathbf{c} \in \mathbb{R}^4$ satisfying all the inequalities listed in (a), the following vector has no negative entries:

$$\mathbf{y}^* = \left(c_2 - c_3, 0, -\frac{1}{3}c_2 + \frac{2}{3}c_3\right).$$

Moreover, it satisfies all the inequalities in the dual problem, with zero surplus in lines 2 and 3. The value of this feasible solution for the dual is

$$g(\mathbf{y}^*) = 7(c_2 - c_3) + 15\left(\frac{2c_3 - c_2}{3}\right) = 2c_2 + 3c_3.$$

(Here we use the vector $\mathbf{b}$ from step (i).) This is the same the value of the feasible solution $\mathbf{x}^* = (0, 2, 3, 0)$ for the primal, so this pair is simultaneously optimal.

2. In the converse, there is no promise of uniqueness for the solution $\mathbf{x}^*$ in the original problem. We can be sure of uniqueness when all the inequalities listed in (a) hold in their strict form, but not in general. See part (c).

(b) As shown above, $\mathbf{b} = (7, 11, 15)$ is the only possibility.

(c) Try $c_3 = 2$. Then the inequality system reduces to

$$2 \leq c_2 \leq 4, \qquad 5c_2 - 2 \geq 3c_1, \qquad 2c_2 + 4 \geq 3c_4.$$

- Taking $c_2 = 2$ satisfies the first condition and reduces the others to

$$8 \geq 3c_1 \quad \text{and} \quad 8 \geq 3c_4.$$

One compatible vector is $\mathbf{c}^{(1)} = (1, 2, 2, 1)$.

- Taking $c_2 = 3$ satisfies the first condition and reduces the others to

$$13 \geq 3c_1 \quad \text{and} \quad 10 \geq 3c_4.$$

A compatible vector is $\mathbf{c}^{(2)} = (1, 3, 2, 1)$.

- Taking $c_2 = 4$ satisfies the first condition and reduces the others to

$$18 \geq 3c_1 \quad \text{and} \quad 12 \geq 3c_4.$$

A compatible vector is $\mathbf{c}^{(3)} = (1, 4, 2, 2)$. Here $f_{\max} = 14$, achieved not only at $\mathbf{x}^*$ but also at $(0, 7/2, 0, 0)$.

Note that $\mathbf{c}^{(2)} - \mathbf{c}^{(1)} = (0, 1, 0, 0)$ and $\mathbf{c}^{(3)} - \mathbf{c}^{(1)} = (0, 2, 0, 1)$ are not parallel, so the three vectors above are not colinear. Each one produces the desired optimum in computerized tests.
[*Sometimes the optimizer is not unique and the computer finds a point different from* $\mathbf{x}^*$. *In such cases it suffices to confirm that* $f(\mathbf{x}^*)$ *agrees with the computed value of* $f_{\max}$.]

3. (a) Some people have memorized this handy formula for the inverse of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Using this is just one of several acceptable ways to show that

$$B = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \quad \Longrightarrow \quad B^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}.$$

(b) The partitioning scheme based on $\mathcal{B} = \{3, 7\}$ and $\mathcal{N} = \{1, 2, 4, 5, 6\}$ puts $Ax = b$ into the form

$$Bx_{\mathcal{B}} + Nx_{\mathcal{N}} = b, \qquad \text{i.e.,} \qquad x_{\mathcal{B}} = B^{-1}b - B^{-1}Nx_{\mathcal{N}}.$$

Here we get

$$\begin{bmatrix} x_3 \\ x_7 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 36 \\ 25 \end{bmatrix} - \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 5 & 6 \\ -1 & 0 & 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 12 & 10 & 13 & 39 & 9 \\ -5 & -4 & -5 & -16 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.$$

Writing these equations explicitly produces the desired dictionary:

$$\begin{aligned} x_3 &= 5 - 12x_1 - 10x_2 - 13x_4 - 39x_5 - 9x_6 \\ x_7 &= 3 + \ 5x_1 + \ 4x_2 + \ 5x_4 + 16x_5 + 3x_6 \end{aligned}$$

(c) The optimality test in the RSM starts with solving $y^T B = c_{\mathcal{B}}^T$. In rare cases like this one, where B^{-1} is known, we can simply write

$$y^T = c_{\mathcal{B}}^T B^{-1} = (\, c_3 \quad c_7 \,) \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix} = (\, 5c_3 - 2c_7 \quad -7c_3 + 3c_7 \,).$$

Then the nonbasic cost coefficient row is

$$\begin{aligned} u^T &= c_{\mathcal{N}}^T - y^T N = (\, c_1 \quad c_2 \quad c_4 \quad c_5 \quad c_6 \,) - (\, 5c_3 - 2c_7 \quad -7c_3 + 3c_7 \,) \begin{pmatrix} 1 & 2 & 4 & 5 & 6 \\ -1 & 0 & 1 & -2 & 3 \end{pmatrix} \\ &= (\, c_1 \quad c_2 \quad c_4 \quad c_5 \quad c_6 \,) \\ &\qquad - (\, (12c_3 - 5c_7) \quad (10c_3 - 4c_7) \quad (13c_3 - 5c_7) \quad (39c_3 - 16c_7) \quad (9c_3 - 3c_7) \,). \end{aligned}$$

The current basis is optimal if and only if each component here is nonpositive. Taking the transpose, we can express this in the requested form:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} \leq \begin{pmatrix} 12 & -5 \\ 10 & -4 \\ 13 & -5 \\ 39 & -16 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} c_3 \\ c_7 \end{pmatrix}.$$

(d) When $c = (1, 3t, 2, 4t, -1, t, 0)$, the system of inequalities above becomes ...

$$\begin{pmatrix} 1 \\ 3t \\ 4t \\ -1 \\ t \end{pmatrix} \leq \begin{pmatrix} 12 & -5 \\ 10 & -4 \\ 13 & -5 \\ 39 & -16 \\ 9 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 24 \\ 20 \\ 26 \\ 78 \\ 18 \end{pmatrix}.$$

All five component inequalities must hold at once. Two are independent of t, and the other three are simultaneously satisfied if and only if $t \leq 13/2$. So

$$\mathcal{B} = \{3, 7\} \text{ is an optimal basis} \qquad \iff \qquad t \leq \frac{13}{2}.$$

4. The difference between the given problem and its antecedent in the textbook is the new variable x_4. Had this variable been present in the original problem, the dictionary in which x_1, w_2, and x_3 are basic variables would have come out like this:

$$\begin{array}{rcl} f & = & 13 - w_1 - 3x_2 - w_3 + \alpha x_4 \\ \hline x_1 & = & 2 - 2w_1 - 2x_2 + w_3 - d_1 x_4 \\ w_2 & = & 1 + 2w_1 + 5x_2 \qquad\quad - d_2 x_4 \\ x_3 & = & 1 + 3w_1 + x_2 - 2w_3 - d_3 x_4 \end{array}$$

To find $\mathbf{d}$ here, recall the dictionary formula $\mathbf{x}_\mathcal{B} = B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_\mathcal{N}$. It shows that $\mathbf{d} = B^{-1}\mathbf{a}^{(4)}$, or $B\mathbf{d} = \mathbf{a}^{(4)}$. So we solve the system

$$\begin{matrix} x_1 & w_2 & x_3 \end{matrix}$$
$$\begin{pmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \text{i.e.,} \qquad \begin{array}{l} 2d_1 \qquad\; + \; d_3 = 1 \\ 4d_1 + d_2 + 2d_3 = 1 \\ 3d_1 \qquad\; + 2d_3 = 1. \end{array}$$

The unique solution here is $(d_1, d_2, d_3) = (1, -1, -1)$, so the lower rows of the new dictionary are

$$\begin{aligned} x_1 &= 2 - 2w_1 - 2x_2 + w_3 - x_4 \\ w_2 &= 1 + 2w_1 + 5x_2 \qquad\quad + x_4 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3 + x_4 \end{aligned}$$

We can use these to refresh the objective function (optionally focussing only on the x_4-dependence):

$$\begin{aligned} f &= 5\,(2 - 2w_1 - 2x_2 + w_3 - x_4) + 4x_2 + 3\,(1 + 3w_1 + x_2 - 2w_3 + x_4) + 3x_4 \\ &= 13 - w_1 - 3x_2 - w_3 + x_4. \end{aligned}$$

Together with the system above, this gives a dictionary that shows x_4 should enter the basis and x_1 should leave. Pivoting with the equation

$$x_4 = 2 - 2w_1 + w_3 - 2x_2 - x_1$$

changes the objective function to

$$f = 15 - x_1 - 5x_2 - 3w_1.$$

Unfortunately the nonbasic variable w_3 has a zero coefficient here, suggesting that the maximizer is not unique. So we need the details of the final dictionary. Here it is:

$$\begin{array}{rcl} f & = & 15 - x_1 - 5x_2 - 3w_1 \\ \hline x_3 & = & 3 - x_1 - x_2 + w_1 - w_3 \\ x_4 & = & 2 - x_1 - 2x_2 - 2w_1 + w_3 \\ w_2 & = & 3 - x_1 + 3x_2 \qquad + w_3 \end{array}$$

We may choose $w_3 = t$ for $0 \le t \le 3$. The full set of solutions is

$$f_{\max} = 15 \qquad \text{attained when} \quad (x_1^*, x_2^*, x_3^*, x_4^*) = (0, 0, 3-t, 2+t)\,, \ 0 \le t \le 3.$$

5. The given problem differs from the one in the textbook because the coefficients of x_3 are all changed. Unfortunately, in the optimal dictionary for the textbook problem, variable x_3 is basic. So we treat this as a problem in which the modified coefficients actually introduce a new variable named $\widetilde{x}_3$, thinking instead about

$$\begin{array}{ll} \text{maximize} & f = 5x_1 + 4x_2 \qquad + 2\widetilde{x}_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 + \widetilde{x}_3 \le 5 \\ & 4x_1 + x_2 + 2x_3 + \widetilde{x}_3 \le 11 \\ & 3x_1 + 4x_2 + 2x_3 + \widetilde{x}_3 \le 8 \\ & x_1, x_2, x_3, \widetilde{x}_3 \ge 0 \end{array}$$

The optimal dictionary for this problem will have the form

$$\begin{array}{rcl} f & = & 13 - w_1 - 3x_2 - w_3 + \alpha\widetilde{x}_3 \\ \hline x_1 & = & 2 - 2w_1 - 2x_2 + w_3 - d_1\widetilde{x}_3 \\ w_2 & = & 1 + 2w_1 + 5x_2 \qquad - d_2\widetilde{x}_3 \\ x_3 & = & 1 + 3w_1 + x_2 - 2w_3 - d_3\widetilde{x}_3 \end{array}$$

To find the mystery coefficients d_1, d_2, d_3, recall that the RSM formulas

$$\mathbf{x}_{\mathcal{B}} = B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_{\mathcal{N}}$$

reveal $\mathbf{d} = B^{-1}\widetilde{\mathbf{a}}^{(3)}$, which is equivalent to $B\mathbf{d} = \widetilde{\mathbf{a}}^{(3)}$, or

$$\overset{\quad x_1 \quad w_2 \quad x_3}{\begin{pmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 0 & 2 \end{pmatrix}} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \text{i.e.,} \qquad \begin{array}{l} 2d_1 \qquad + d_3 = 1 \\ 4d_1 + d_2 + 2d_3 = 1\,. \\ 3d_1 \qquad + 2d_3 = 1 \end{array}$$

The unique solution here is $(d_1, d_2, d_3) = (1, -1, -1)$, leading to the lower equations

$$\begin{array}{rcl} x_1 & = & 2 - 2w_1 - 2x_2 + w_3 - \widetilde{x}_3 \\ w_2 & = & 1 + 2w_1 + 5x_2 \qquad + \widetilde{x}_3 \\ x_3 & = & 1 + 3w_1 + x_2 - 2w_3 + \widetilde{x}_3 \end{array}$$

Now it is $\widetilde{x}_3$ that should appear in the basis, so we pivot it in and x_3 out, writing

$$\widetilde{x}_3 = -1 - x_2 + x_3 - 3w_1 + 2w_3.$$

Back-substitution gives

$$\begin{array}{rcl} x_1 & = & 3 - x_2 - x_3 + w_1 - w_3 \\ \widetilde{x}_3 & = & -1 - x_2 + x_3 - 3w_1 + 2w_3 \\ w_2 & = & 0 + 4x_2 + x_3 - w_1 + 2w_3 \end{array}$$

In the modified objective, these choices imply

$$f = 5x_1 + 4x_2 + 2\widetilde{x}_3 = 13 - 3x_2 - 3x_3 - w_1 - w_3.$$

Now we have no further use for the variable x_3, which has become nonbasic, so we do not write it in the dictionary summarizing our work so far:

$$\begin{array}{rcl} f & = & 13 - 3x_2 - \ w_1 - \ w_3 \\ \hline x_1 & = & 3 - \ x_2 + \ w_1 - \ w_3 \\ \widetilde{x}_3 & = & -1 - \ x_2 - 3w_1 + 2w_3 \\ w_2 & = & 0 + 4x_2 - \ w_1 + 2w_3 \end{array}$$

This dictionary is dual-feasible, although not feasible, so we proceed with the Dual Simplex Method. Variable $\widetilde{x}_3$ must leave the basis, and the entering variable is found by considering

$$f + t\widetilde{x}_3 = (13 - t) - (3 + t)x_2 - (1 + 3t)w_1 - (1 - 2t)w_3.$$

As $t \geq 0$ increases, w_3 first attracts a zero coefficient, so w_3 must enter the basis. We pivot w_3 with $\widetilde{x}_3$ via

$$w_3 = \frac{1}{2} + \frac{1}{2}x_2 + \frac{1}{2}\widetilde{x}_3 + \frac{3}{2}w_1.$$

Substitution gives an optimal dictionary:

$$\begin{array}{rcl} f & = & 12.5 - 3.5x_2 - 0.5\widetilde{x}_3 - 2.5w_1 \\ \hline x_1 & = & 2.5 - 1.5x_2 - 0.5\widetilde{x}_3 - 0.5w_1 \\ w_2 & = & 1.0 + 5.0x_2 + \ \widetilde{x}_3 + 2.0w_1 \\ w_3 & = & 0.5 + 0.5x_2 + 0.5\widetilde{x}_3 + 1.5w_1 \end{array}$$

The modified problem has $f_{\max} = 12.5$, attained uniquely by $(x_1^*, x_2^*, \widetilde{x}_3^*) = (2.5, 0, 0)$.

6. (a) Suppose the firm's daily production is

$$x_1 \text{ bookcases}, \quad x_2 \text{ desks}, \quad x_3 \text{ chairs}, \quad \text{and} \quad x_4 \text{ futon frames}.$$

Of course, each $x_i \geq 0$. Each product generates a different net profit; the firm's total profit is

$$f = 19x_1 + 13x_2 + 12x_3 + 17x_4.$$

Comparing the materials needed to produce each product with the amounts available leads to the three constraints in the firm's optimization problem, which is to

$$\begin{array}{lll} \text{maximize} & f = 19x_1 + 13x_2 + 12x_3 + 17x_4 & \\ \text{subject to} & 3x_1 + 2x_2 + x_3 + 2x_4 \leq 225 & \text{(hours of work)} \\ & x_1 + x_2 + x_3 + x_4 \leq 117 & \text{(units of metal)} \\ & 4x_1 + 3x_2 + 3x_3 + 4x_4 \leq 420 & \text{(units of wood)} \\ & x_1, x_2, x_3, x_4 \geq 0 & \end{array}$$

The optimal production levels are given: $x_1^* = 39$, $x_2^* = 0$, $x_3^* = 48$, $x_4^* = 30$. Since a basis in this problem must have exactly three elements, we deduce that $\mathcal{B} = \{1, 3, 4\}$ is the optimal basis for the current situation. To find the corresponding dictionary, we need to note that

$$B = \overset{\begin{array}{ccc} x_1 & x_3 & x_4 \end{array}}{\begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 3 & 4 \end{pmatrix}}, \quad \text{implies} \quad B^{-1} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 4 & -1 \\ -1 & -5 & 2 \end{pmatrix}.$$

If we write $(x_5, x_6, x_7) = (w_1, w_2, w_3)$ for the slack variables above, the lower rows of the optimal dictionary are $x_{\mathcal{B}} = B^{-1}b - B^{-1}Nx_{\mathcal{N}}$, i.e.,

$$\begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 4 & -1 \\ -1 & -5 & 2 \end{pmatrix} \begin{bmatrix} 225 \\ 117 \\ 420 \end{bmatrix} - \begin{pmatrix} 1 & 2 & -1 \\ 0 & 4 & -1 \\ -1 & -5 & 2 \end{pmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 39 \\ 48 \\ 30 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 0 & 4 & -1 \\ -1 & -1 & -5 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$$

The famous RSM formulas also specify the objective row. The optimal dictionary is

$$\begin{array}{rl} f & = 1827 - x_2 - 2x_5 - x_6 - 3x_7 \\ \hline x_1 = & 39 - x_2 - x_5 - 2x_6 + x_7 \\ x_3 = & 48 - x_2 \qquad\quad - 4x_6 + x_7 \\ x_4 = & 30 + x_2 + x_5 + 5x_6 - 2x_7 \end{array}$$

(b) Boosting the amount of metal available from 117 to 125 changes the second component of the constraint vector b. This affects the numbers in the current BFS: the new ones obey

$$\begin{matrix} x_1 & x_3 & x_4 \end{matrix}$$
$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 225 \\ 125 \\ 420 \end{pmatrix}, \qquad \text{i.e.,} \qquad \begin{aligned} 3x_1 + x_3 + 2x_4 &= 225 \\ x_1 + x_3 + x_4 &= 125 \\ 4x_1 + 3x_3 + 4x_4 &= 420 \end{aligned}$$

The unique solution ($x_{\mathcal{B}}^* = B^{-1}b$) is $x_1^* = 55$, $x_3^* = 80$, $x_4^* = -10$. Its objective value is

$$f = 19x_1^* + 13x_2^* + 12x_3^* + 17x_4^* = 1835.$$

So a dual-feasible dictionary for the new problem is

$$\begin{array}{rl} f & = 1835 - x_2 - 2x_5 - x_6 - 3x_7 \\ \hline x_1 = & 55 - x_2 - x_5 - 2x_6 + x_7 \\ x_3 = & 80 - x_2 \qquad\quad - 4x_6 + x_7 \\ x_4 = & -10 + x_2 + x_5 + 5x_6 - 2x_7 \end{array}$$

This dictionary is infeasible: thinking simplistically about how to use the additional metal produces a scheme we can't implement without having more wood as well. But we don't, so we need to think more strategically. The infeasibility identifies x_4 as a leaving variable. Expressing

$$\begin{aligned} f + tx_4 &= 1835 - x_2 - 2x_5 - x_6 - 3x_7 + t\,(-10 + x_2 + x_5 + 5x_6 - 2x_7) \\ &= (1835 - 10t) + (t-1)x_2 + (t-2)x_5 + (5t-1)x_6 - (2t+3)x_7 \end{aligned}$$

shows that x_6 should enter in place of x_4 (and $t^* = 1/5$ gives a preview of the objective row when it does). Pivoting gives the optimal dictionary

$$\begin{array}{rl} f & = 1833 - 0.8x_2 - 1.8x_5 - 0.2x_4 - 3.4x_7 \\ \hline x_1 = & 51 - 0.6x_2 - 0.6x_5 - 0.4x_4 + 0.2x_7 \\ x_3 = & 72 - 0.2x_2 + 0.8x_5 - 0.8x_4 + -0.6x_7 \\ x_6 = & 2 - 0.2x_2 - 0.2x_5 + 0.2x_4 + 0.4x_7 \end{array}$$

All the company's production is now devoted to 51 bookcases and 72 chairs per day; its profit rises to \$1833.

(c) We can express the new requirement that $x_3 \leq 5x_2$ by defining a new slack variable x_8:

$$x_8 = 5x_2 - x_3, \quad \text{with} \quad x_8 \geq 0.$$

We add this definition to the bottom of our original optimal dictionary, effectively adding x_8 to the list of basic variables. To fit the dictionary, the right side must be expressed using only nonbasic variables. So we calculate

$$x_8 = 5x_2 - x_3 = 5x_2 - (48 - x_2 - 4x_6 + x_7) = -48 + 6x_2 + 4x_6 - x_7.$$

Since the new variable has a coefficient of 0 in the cost, there is no impact on the objective function. A dual-feasible dictionary is

$$\begin{array}{rcl}
f &=& 1827 - x_2 - 2x_5 - x_6 - 3x_7 \\
\hline
x_1 &=& 39 - x_2 - x_5 - 2x_6 + x_7 \\
x_3 &=& 48 - x_2 \qquad\quad - 4x_6 + x_7 \\
x_4 &=& 30 + x_2 + x_5 + 5x_6 - 2x_7 \\
x_8 &=& -48 + 6x_2 \qquad + 4x_6 - x_7
\end{array}$$

Again the Dual Simplex Method is appropriate. Arranging primal feasibility will require x_8 to leave the basis. Notice

$$\begin{aligned}
f + tx_8 &= 1827 - x_2 - 2x_5 - x_6 - 3x_7 + t\,(-48 + 6x_2 + 4x_6 - x_7 - s) \\
&= (1827 - 48t) + (6t - 1)x_2 - 2x_5 + (4t - 1)x_6 - (t + 3)x_7.
\end{aligned}$$

Here $t = 1/6$ is largest choice that maintains dual feasibility. It identifies x_2 as the entering variable. Pivoting leads to

$$\begin{array}{rcl}
f &=& 1819 - (1/6)x_8 - 2x_5 - (1/3)x_6 - (19/6)x_7 \\
\hline
x_1 &=& 31 - (1/6)x_8 - x_5 - (4/3)x_6 + (5/6)x_7 \\
x_3 &=& 40 - (1/6)x_8 \qquad - (10/3)x_6 + (5/6)x_7 \\
x_4 &=& 38 + (1/6)x_8 + x_5 + (13/3)x_6 - (11/6)x_7 \\
x_2 &=& 8 + (1/6)x_8 \qquad - (2/3)x_6 + (1/6)x_7
\end{array}$$

This optimal dictionary reveals a net profit of \$1819 (which is lower than the \$1827 we had in the original problem, because new limitations can never improve profits), with a production plan of

$$x_1 = 31 \text{ bookcases}, \quad x_2 = 8 \text{ desks}, \quad x_3 = 40 \text{ chairs}, \quad x_4 = 38 \text{ bedframes}.$$