Math 340(921) Problem Set 6

Due in class on Thursday 20 June 2013 (or by 14:00 on Friday 21 Jun)

1. The "Diet Problem" is a classic LP application. The idea is to minimize the cost of adequate nutrition, given a collection of different foods and some knowledge of the nutritional elements in each one. Entertaining stories about who is being nourished and what they are being forced to eat are everywhere. A no-frills example is "Model 2: Pig Farming" on the page

`http://www.math.washington.edu/~burke/crs/407/models/`

(The same page contains many other interesting practical problems.)

(a) Let g denote the cost in cents when each pig is fed y_1 kilograms of corn, y_2 kilos of tankage, and y_3 kilos of alfalfa. Set up the LP for minimum feed cost subject to the nutritional requirements.

(b) Find a primal-form LP for which the problem written in part (a) is the dual.

(c) Use a computer to solve the primal-form problem stated in (b). Print your results and, without further reference to the computer, show how to extract the solution to the original problem in (a). Then confirm your findings by having the computer solve the problem in (a) directly.

(d) Write an economic interpretation for each component of the maximizing vector for the problem stated in (b).

2. It's your turn to invent a homework problem. You need a standard-form LP like this:

$$\begin{aligned}
\text{maximize } g &= c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{subject to } \quad & 3x_1 + 2x_2 + x_3 + 2x_4 \le b_1 \\
& x_1 + x_2 + x_3 + x_4 \le b_2 \\
& 4x_1 + 3x_2 + 3x_3 + 4x_4 \le b_3 \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}$$

You want x_2, x_3, and w_2 to be basic in the optimal configuration, so you decide that an optimal basic feasible solution (BFS) should give positive values to these three variables.

(a) Prove that $x^* = (0, 2, 3, 0)$ is an optimal BFS with slack $w^* = (0, 6, 0)$ iff $c \in \mathbb{R}^4$ obeys

$$c_3 \le c_2 \le 2c_3, \quad \text{and} \quad 5c_2 - c_3 \ge 3c_1, \quad \text{and} \quad 2c_2 + 2c_3 \ge 3c_4.$$

(b) What choices of $b \in \mathbb{R}^3$ are compatible with the situation in part (a)?

(c) Find three nonzero vectors c compatible with the requirements in part (a). Make sure they do not all lie on the same line in $\mathbb{R}^4$. Use a computerized LP solver to confirm statement (a) for each of your three choices.

3. (a) Find the inverse of this matrix:

$$B = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}.$$

(b) Use your answer from part (a) to express the linear system $Ax = b$ in the form of a dictionary with x_3 and x_7 basic, given

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -1 & 0 & 2 & 1 & -2 & 3 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} 36 \\ 25 \end{bmatrix}.$$

(c) Taking A and b from part (b), write a system of 5 linear inequalities involving the components of $c \in \mathbb{R}^7$ that identifies precisely when $\mathcal{B} = \{3, 7\}$ is an optimal basis for the problem

$$\max \left\{ c^T x \,:\, Ax = b,\ x \ge 0 \right\}. \tag{$*$}$$

More . . .

Present your answer in the form below, where M is a matrix that you identify explicitly:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} \leq M \begin{bmatrix} c_3 \\ c_7 \end{bmatrix}.$$

(d) Suppose $c = (1, 3t, 2, 4t, -1, t, 0)$ in $(*)$. Find all real t for which $\mathcal{B} = \{3, 7\}$ is an optimal basis.

4. Consider the textbook's first example of a standard LP, shown in line (2.1) of page 13. After a step-by-step introduction to the reasoning behind the simplex method, the author presents the optimal dictionary for this problem in line (2.10) on page 17. Use this information to solve the following problem by doing some preliminary work and then making only one pivot:

$$\begin{array}{ll} \text{maximize} & f = 5x_1 + 4x_2 + 3x_3 + 3x_4 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 + x_4 \leq 5 \\ & 4x_1 + x_2 + 2x_3 + x_4 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 + x_4 \leq 8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

5. The problem below is quite similar to the standard LP shown in line (2.1) of our textbook, on page 13. Note that the optimal dictionary for the textbook problem is given on page 17. Use this information to solve the following problem with at most two pivots (including preliminary work):

$$\begin{array}{ll} \text{maximize} & f = 5x_1 + 4x_2 + 2x_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + x_3 \leq 11 \\ & 3x_1 + 4x_2 + x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

6. Wenceslas Woodworks currently makes bookcases, desks, chairs, and futon frames. Each day there are 225 hours of labour, 117 units of metal, and 420 units of wood available for use in production.

- A bookcase uses three hours of work, one unit of metal, and four units of wood.
- A desk uses two hours of work, one unit of metal, and three units of wood.
- A chair uses one hour of work, one unit of metal, and three units of wood.
- A futon frame uses two hours of work, one unit of metal, and four units of wood.

The net profit per item is $19 for a bookcase, $13 for a desk, $12 for a chair, and $17 for a futon frame. The profit-maximizing production plan is to make 39 bookcases, 48 chairs, and 30 futon frames each day.

(a) Express the problem of maximizing profit as a standard LP.

Find the optimal strategy when the following changes are made, *independently,* to the original setup.

(b) The availability of metal increases from 117 to 125 units each day.

(c) The number of chairs produced cannot be more than five times the number of desks.

[These are excerpts from Problem 10.2 in Chvátal's fine textbook on Linear Programming.]