M340(921) Solutions—Problem Set 5

1. Rearranging the first constraint puts this problem into a form that is convenient to dualize:

$$\begin{array}{lrl} \text{maximize} & \zeta = & x_1 - 2x_2 - 3x_3 \\ \text{subject to} & & -x_2 - 2x_3 \le -1 \\ & & x_1 + 3x_3 \le 2 \\ & & 2x_1 - 3x_2 = 3 \\ & & x_1, x_2 \ge 0,\ x_3 \text{ unrestricted} \end{array}$$

The relations in the three primal constraint rows dualize to produce the sign conditions on the dual variables:

$$y_1 \ge 0, \qquad y_2 \ge 0, \qquad y_3 \in \mathbb{R}.$$

The primal sign conditions $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \in \mathbb{R}$ dualize to produce the relations in the dual constraints:

$$0y_1 + y_2 + 2y_3 \ge 1, \qquad -y_1 + 0y_2 - 3y_3 \ge -2, \qquad -2y_1 + 3y_2 + 0y_3 = -3.$$

Writing the dual objective and collecting the constraints, we arrive at the dual problem

$$\begin{array}{lrl} \text{minimize} & \xi = & -y_1 + 2y_2 + 3y_3 \\ \text{subject to} & & y_2 + 2y_3 \ge 1 \\ & & -y_1 - 3y_3 \ge -2 \\ & & -2y_1 + 3y_2 = -3 \\ & & y_1, y_2 \ge 0,\ y_3 \text{ unrestricted} \end{array}$$

Except for a change in names between x_j and y_j, and an equivalent presentation of each constraint based on multiplication by -1, this is identical to the problem given in the first place!

2. The given LP is

$$\begin{array}{lrl} \text{maximize} & f = & 2x_1 - 6x_2 \\ \text{subject to} & & -x_1 - x_2 - x_3 \le -2 \\ & & 2x_1 - x_2 + x_3 \le 1 \\ & & x_1, x_2, x_3 \ge 0 \end{array}$$

The "Dual Phase I" step starts by replacing the given objective f with one that is obviously dual-feasible. Any choice involving only negative coefficients will work; we follow the suggestion and opt for $\widetilde{f} = -x_1 - x_2 - x_3$. With the usual notation for primal slack variables, the dictionary for the modified problem is

$$\begin{array}{rcl} \widetilde{f} & = & - x_1 - x_2 - x_3 \\ \hline w_1 & = & -2 + x_1 + x_2 + x_3 \\ w_2 & = & 1 - 2x_1 + x_2 - x_3 \end{array}$$

Select Pivot 1 [Dual Simplex Method]. The most infeasible row involves w_1, so variable w_1 must leave the basis. To make this happen, consider the sum

$$\widetilde{f} + tw_1 = -2t + (t-1)x_1 + (t-1)x_2 + (t-1)x_3.$$

The smallest $t > 0$ for which one of the coefficients becomes 0 is $t = 1$, and it reveals each of x_1, x_2, x_3 as a candidate to disappear from the objective row and enter the basis. The one we choose will take on the value 2, and this seems to favour x_2 because of the positive coefficients on x_2 in the lower rows of the dictionary. So let's use x_2 as the entering variable.

Execute Pivot 1. To pivot x_2 into the basis and w_1 out, use $x_2 = 2 - x_1 - x_3 + w_1$. Updating gives

$$\begin{array}{rl} \widetilde{f} &= -2 \qquad\qquad\quad - w_1 \\ \hline x_2 = & 2 - \ x_1 - \ x_3 + w_1 \\ w_2 = & 3 - 3x_1 - 2x_3 + w_1 \end{array}$$

This is optimal for the modified problem.

Restore Objective. This dictionary above has the wrong objective, but at least it is feasible. We restore the true objective by substituting from the dictionary's lower rows:

$$f = 2x_1 - 6x_2 = 2x_1 - 6(2 - x_1 - x_3 + w_1) = -12 + 8x_1 + 6x_3 - 6w_1.$$

A feasible dictionary for the original problem is

$$\begin{array}{rl} f &= -12 + 8x_1 + 6x_3 - 6w_1 \\ \hline x_2 = & \quad 2 - \ x_1 - \ x_3 + \ w_1 \\ w_2 = & \quad 3 - 3x_1 - 2x_3 + \ w_1 \end{array}$$

Select Pivot 2 [Primal Simplex Method]. Choose x_1 to enter because it has the largest coefficient, and x_3 to leave in order to maintain feasibility.

Execute Pivot 2. The pivot equation is $x_1 = 1 - (2/3)x_3 + (1/3)w_1 - (1/3)w_2$. It gives

$$\begin{array}{rl} f &= -4 + (2/3)x_3 - (10/3)w_1 - (8/3)w_2 \\ \hline x_1 = & \quad 1 - (2/3)x_3 + \ (1/3)w_1 - (1/3)w_2 \\ x_2 = & \quad 1 - (1/3)x_3 + \ (2/3)w_1 + (1/3)w_2 \end{array}$$

Select Pivot 3 [PSM]. Choose x_3 to enter because it has the only positive coefficient, and x_1 to leave in order to maintain feasibility.

Execute Pivot 3. The pivot equation is $x_3 = (3/2) - (3/2)x_1 + (1/2)w_1 - (1/2)w_2$. It gives

$$\begin{array}{rl} f &= \ -3 - \qquad x_1 - \qquad 3w_1 - \qquad 3w_2 \\ \hline x_2 = & (1/2) + (1/2)x_1 + (1/2)w_1 + (1/2)w_2 \\ x_3 = & (3/2) - (3/2)x_1 + (1/2)w_1 - (1/2)w_2 \end{array}$$

Conclusion. This is an optimal dictionary for the original problem. In terms of the original variables, $f_{\max} = -3$, achieved uniquely at $x^* = (0, 1/2, 3/2)$.

Check (not required for credit). Note that $y^* = (3, 3)$ is feasible in the dual problem, where it gives $g_{\min} = -3 = f_{\max}$. This confirms the optimality of x^*. The dual problem is

$$\begin{array}{ll} \text{minimize } g = & -2y_1 + \ y_2 \\ \text{subject to} & -y_1 + 2y_2 \geq \ \ 2 \\ & -y_1 - \ y_2 \geq -6 \\ & -y_1 + \ y_2 \geq \ \ 0 \\ & \ \ y_1, y_2 \geq 0 \end{array}$$

3. (a)(b) In the requested two-column presentation below, the first few primal dictionaries are not optimal, while the first few dual dictionaries are not feasible. Each pivot brings optimality and feasibility closer together; both processes produce an optimal final dictionary after two pivots.

Initial dictionary (primal):

$$\begin{array}{rcl} f &=& 4x_1 + 5x_2 \\ \hline w_1 &=& 5 - x_1 - x_2 \\ w_2 &=& 8 - x_1 - 2x_2 \\ w_3 &=& 9 - 2x_1 - x_2 \end{array}$$

Current basic variables in bold boxes:

$$x_1, x_2, \boxed{\mathbf{w_1}}, \boxed{\mathbf{w_2}}, \boxed{\mathbf{w_3}}.$$

We select x_2 to enter, w_2 to leave.

Initial dictionary (dual):

$$\begin{array}{rcl} -g &=& -5y_1 - 8y_2 - 9y_3 \\ \hline z_1 &=& -4 + y_1 + y_2 + 2y_3 \\ z_2 &=& -5 + y_1 + 2y_2 + y_3 \end{array}$$

Current basic variables in bold boxes:

$$\boxed{\mathbf{z_1}}, \boxed{\mathbf{z_2}}, y_1, y_2, y_3.$$

The primal selections make z_2 leave, y_2 enter.

Primal dictionary after 1 pivot:

$$\begin{array}{rcl} f &=& 20 + \frac{3}{2}x_1 - \frac{5}{2}w_2 \\ \hline x_2 &=& 4 - \frac{1}{2}x_1 - \frac{1}{2}w_2 \\ w_1 &=& 1 - \frac{1}{2}x_1 + \frac{1}{2}w_2 \\ w_3 &=& 5 - \frac{3}{2}x_1 + \frac{1}{2}w_2 \end{array}$$

Current basic variables in bold boxes:

$$x_1, \boxed{\mathbf{x_2}}, \boxed{\mathbf{w_1}}, w_2, \boxed{\mathbf{w_3}}.$$

We select x_1 to enter, w_1 to leave.

Dual dictionary after 1 pivot:

$$\begin{array}{rcl} -g &=& -20 - 4z_2 - y_1 - 5y_3 \\ \hline z_1 &=& -\frac{3}{2} + \frac{1}{2}z_2 + \frac{1}{2}y_1 + \frac{3}{2}y_3 \\ y_2 &=& \frac{5}{2} + \frac{1}{2}z_2 - \frac{1}{2}y_1 - \frac{1}{2}y_3 \end{array}$$

Current basic variables in bold boxes:

$$\boxed{\mathbf{z_1}}, z_2, y_1, \boxed{\mathbf{y_2}}, y_3.$$

The primal selections make z_1 leave, y_1 enter.

Primal dictionary after 2 pivots:

$$\begin{array}{rcl} f &=& 23 - 3w_1 - w_2 \\ \hline x_1 &=& 2 - 2w_1 + w_2 \\ x_2 &=& 3 + w_1 - w_2 \\ w_3 &=& 2 + 3w_1 - w_2 \end{array}$$

Current basic variables in bold boxes:

$$\boxed{\mathbf{x_1}}, \boxed{\mathbf{x_2}}, w_1, w_2, \boxed{\mathbf{w_3}}.$$

This is an optimal dictionary.

Dual dictionary after 2 pivots:

$$\begin{array}{rcl} -g &=& -23 - 2z_1 - 3z_2 - 2y_3 \\ \hline y_1 &=& 3 + 2z_1 - z_2 - 3y_3 \\ y_2 &=& 1 - z_1 + z_2 + y_3 \end{array}$$

Current basic variables in bold boxes:

$$z_1, z_2, \boxed{\mathbf{y_1}}, \boxed{\mathbf{y_2}}, y_3.$$

This is an optimal dictionary.

(c) To translate the primal dictionary into the dual dictionary directly, sort the variables appearing in both rows and columns of both dictionaries to appear with increasing indices. Then just take the negative-transpose of one set of coefficients to produce the complementary set.

Sorting the variables is essential to make the simple "negative transpose" idea work. If the variables get scrambled while pivoting, we need to be more precise: the equations for a basic variable like y_2 in the dual dictionary must come from the column associated with its twin nonbasic variable (here w_2) on the primal side. Each coefficient in that column is attached to a particular basic primal variable, and its negation becomes the coefficient for the associated nonbasic dual variable.

(d) Working from right to left with the recipe articulated above gives the following correspondence:

$$\begin{array}{rcl} f &=& 18 + 3x_2 - 2w_3 \\ \hline x_1 &=& (9/2) - (1/2)x_2 - (1/2)w_3 \\ w_1 &=& (1/2) - (1/2)x_2 + (1/2)w_3 \\ w_2 &=& (7/2) - (3/2)x_2 + (1/2)w_3 \end{array}$$

$$\begin{array}{rcl} -g &=& -18 - (9/2)z_1 - (1/2)y_1 - (7/2)y_2 \\ \hline z_2 &=& -3 + (1/2)z_1 + (1/2)y_1 + (3/2)y_2 \\ y_3 &=& 2 + (1/2)z_1 - (1/2)y_1 - (1/2)y_2 \end{array}$$

4. (a) Let's temporarily use w_j as the symbol for the slack in inequality j. Then the proposed change is to replace the original constraint

$$w_2 \geq 0, \quad \text{where} \quad w_2 = 11 - 4x_1 - x_2 - 2x_3,$$

with the new constraint

$$\widetilde{w}_2 \geq 0, \quad \text{where} \quad \widetilde{w}_2 = 4 - 4x_1 - x_2 - 2x_3.$$

Since $w_2 - \widetilde{w}_2 = 7$ is a constant independent of all other variables in the problem, we can replace w_2 by $\widetilde{w}_2 + 7$ everywhere in the optimal dictionary. A slight rearrangement then gives this dual-feasible dictionary for the modified problem:

$$\begin{array}{rcl} f &=& 13 - 3x_2 - w_1 - w_3 \\ \hline x_1 &=& 2 - 2x_2 - 2w_1 + w_3 \\ x_3 &=& 1 + x_2 + 3w_1 - 2w_3 \\ \widetilde{w}_2 &=& -6 + 5x_2 + 2w_1 \end{array}$$

Now it simplifies the notation to define $x_4 = w_1$, $x_5 = \widetilde{w}_2$, $x_6 = w_3$, and express feasibility in the modified problem through the simple componentwise inequality $\mathbf{x} \geq \mathbf{0}$ in $\mathbb{R}^6$. The dictionary now looks like this:

$$\begin{array}{rcl} f &=& 13 - 3x_2 - x_4 - x_6 \\ \hline x_1 &=& 2 - 2x_2 - 2x_4 + x_6 \\ x_3 &=& 1 + x_2 + 3x_4 - 2x_6 \\ x_5 &=& -6 + 5x_2 + 2x_4 \end{array}$$

It is not primal-feasible because $x_5^* < 0$. So x_5 must leave the basis. To identify the entering variable, use the equation involving x_5 to write

$$f + tx_5 = (13 - 6t) - (3 - 5t)x_2 - (1 - 2t)x_4 - x_6.$$

The smallest $t \geq 0$ that introduces a zero coefficient on the right is $t^* = 1/2$: the zero coefficient applies to x_4, so x_4 will enter the basis. We pivot as usual to interchange x_5 and x_4, using the equation

$$x_4 = 3 - \frac{5}{2}x_2 + \frac{1}{2}x_5,$$

to arrive at

$$\begin{array}{rcl} f &=& 10 - 0.5x_2 - 0.5x_5 - x_6 \\ \hline x_1 &=& -4 + 3.0x_2 - 1.0x_5 + x_6 \\ x_3 &=& 10 - 6.5x_2 + 1.5x_5 - 2x_6 \\ x_4 &=& 3 - 2.5x_2 + 0.5x_5 \end{array}$$

This is still infeasible, as shown by the row involving x_1. So x_1 must leave the basis. To decide which variable enters, use the equation involving x_1 to express

$$f + tx_1 = (10 - 4t) - (\tfrac{1}{2} - 3t)x_2 - (\tfrac{1}{2} + t)x_5 - (1 - t)x_6 - tx_1.$$

The smallest $t \geq 0$ to produce a zero coefficient on the right is $t^* = 1/6$: it marks x_2 as the unique candidate to enter the basis. So we pivot as usual to interchange x_1 and x_2, using the equation

$$x_2 = \frac{4}{3} + \frac{1}{3}x_1 + \frac{1}{3}x_5 - \frac{1}{3}x_6$$

to arrive at

$$\begin{array}{rl} f \;= & (28/3) - \;\;(1/6)x_1 - (2/3)x_5 - (5/6)x_6 \\ \hline x_2 = & \;\;(4/3) + \;\;(1/3)x_1 + (1/3)x_5 - (1/3)x_6 \\ x_3 = & \;\;(4/3) - (13/6)x_1 - (2/3)x_5 + (1/6)x_6 \\ x_4 = & -(1/3) - \;\;(5/6)x_1 - (1/3)x_5 + (5/6)x_6 \end{array}$$

This is still infeasible, as shown by the row involving x_4. So x_4 must leave the basis. To decide which variable enters, use the equation involving x_4 to express

$$f + tx_4 = \left(\frac{28-t}{3}\right) - \left(\frac{1+5t}{6}\right)x_1 - \left(\frac{2+t}{3}\right)x_5 - \left(\frac{5-5t}{6}\right)x_6.$$

The smallest $t \geq 0$ to produce a zero coefficient on the right is $t^* = 1$: it marks x_6 to enter the basis. So we pivot as usual to interchange x_4 and x_6, arriving at

$$\begin{array}{rl} f & = 9.0 - \;\; x_1 - 1.0x_4 - 1.0x_5 \\ \hline x_2 & = 1.2 \qquad\quad - 0.4x_4 + 0.2x_5 \\ x_3 & = 1.4 - 2x_1 + 0.2x_4 - 0.6x_5 \\ x_6 & = 0.4 + \;\; x_1 + 1.2x_4 + 0.4x_5 \end{array}$$

At last, this dictionary is optimal. The (unique) primal maximizer is $(0, 1.2, 1.4)$; the maximum value is 9.

(b) (i) When $\theta > 0$, the new constraint requires $4x_1 + x_2 + 2x_3 < 0$. But the problem requires $4x_1 + x_2 + 2x_3 \geq 0$. No numbers x_1, x_2, x_3 can make both statements true.

(ii) We can proceed as in part (a). The proposed replacement constraint is

$$\widehat{w}_2 \geq 0, \quad \text{where} \quad \widehat{w}_2 = -\theta - 4x_1 - x_2 - 2x_3.$$

In relation to the original slack variable w_2 (see part (a)), we have $w_2 - \widehat{w}_2 = 11 + \theta$, so we can write the dictionary in terms of the new slack variable by replacing w_2 with $\widehat{w}_2 + 11 + \theta$ and rearranging slightly. With x_5 now denoting the new $\widehat{w}_2$, we have

$$\begin{array}{rr} f \;= & 13 - 3x_2 - \;\; x_4 - \;\; x_6 \\ \hline x_1 = & 2 - 2x_2 - 2x_4 + \;\; x_6 \\ x_3 = & 1 + \;\; x_2 + 3x_4 - 2x_6 \\ x_5 = & -(10+\theta) + 5x_2 + 2x_4 \qquad\quad \end{array} \tag{$\dagger$}$$

This is not primal-feasible because $x_5^* < 0$, but the dictionary is identical to the one in part (a) in every way except for the exact value of x_5^*. So let's guess that the choice of basis that was optimal in part (a) could produce a useful dictionary here as well. The pivot steps in (a) will produce the same entries in all columns except the first one, so the dictionary just shown will transform into one having the form

$$\begin{array}{rl} f & = f^* - \;\; x_1 - 1.0x_4 - 1.0x_5 \\ \hline x_2 & = x_2^* \qquad\quad - 0.4x_4 + 0.2x_5 \\ x_3 & = x_3^* - 2x_1 + 0.2x_4 - 0.6x_5 \\ x_6 & = x_6^* + \;\; x_1 + 1.2x_4 + 0.4x_5 \end{array}$$

When we get this dictionary, plugging $x_1 = 0$, $x_4 = 0$, and $x_5 = 0$ into it will reveal $x_2 = x_2^*$, etc. But since all dictionaries in a given problem encode identical information, we can make these substitutions in (†) instead:

$$\begin{array}{rcl} f & = & 13 - 3x_2 - \ \ x_6 \\ \hline 0 & = & 2 - 2x_2 + \ \ x_6 \\ x_3 & = & 1 + \ \ x_2 - 2x_6 \\ 0 & = & -(10+\theta) + 5x_2 \end{array} \tag{†}$$

This shows $x_2 = 2 + \theta/5$, then $x_6 = 2 + 2\theta/5$, then $x_3 = -1 - 3\theta/5$, and $f = 5 - \theta$. We arrive at the dual-feasible dictionary

$$\begin{array}{rcl} f & = & (5-\theta) - \ \ x_1 - 1.0x_4 - 1.0x_5 \\ \hline x_2 & = & (10+\theta)/5 \qquad\quad - 0.4x_4 + 0.2x_5 \\ x_3 & = & -(5+3\theta)/5 - 2x_1 + 0.2x_4 - 0.6x_5 \\ x_6 & = & (10+2\theta)/5 + \ \ x_1 + 1.2x_4 + 0.4x_5 \end{array}$$

This is still not feasible, so we try a DSM step to make x_3 leave the basis:

$$f + tx_3 = 5 - \theta - t(5+3\theta)/5 - (1+2t)x_1 - (1-t/5)x_4 - (1+3t/5)x_5.$$

This selects x_4 as the entering variable, and produces the pivot equation

$$x_4 = (5+3\theta) + 10x_1 + 5x_3 + 3x_5.$$

In the first step of back-substitution, we find

$$x_2 = -\theta - 4x_1 - 2x_3 - x_5.$$

Here, at last is the disaster we seek. Whenever $\theta > 0$, any feasible input would have to make the right side negative and the left side nonnegative. This is impossible, so no feasible inputs exist! It's safe to stop here, but to see the catastrophe in slow motion, look at the fully updated dictionary:

$$\begin{array}{rcl} f & = & -4\theta - 11x_1 - 5x_3 - 4x_5 \\ \hline x_2 & = & -\theta - \ \ 4x_1 - 2x_3 - \ \ x_5 \\ x_4 & = & (5+3\theta) + 10x_1 + 5x_3 + 3x_5 \\ x_6 & = & (8+4\theta) + 13x_1 + 6x_3 + 4x_5 \end{array}$$

Here $x_2^* < 0$ suggests a dual-simplex pivot, so we form

$$f + tx_2 = -(4+t)\theta - (11+t)x_1 - (5+2t)x_3 - (4+t)x_5.$$

There is no choice of $t \geq 0$ that will create a zero coefficient on the right. This means that there is no way to find a basic feasible solution for the original system. This completes the proof.

(c) Constants with the specified property do not exist. To see why, pick any arbitrary numbers α, β, δ and name the associated problem (P). Consider the dual of (P), namely,

$$\begin{array}{rl} \text{Minimize} & \alpha y_1 + \beta y_2 + \delta y_3 \\ \text{Subject to} & 2y_1 + 4y_2 + 3y_3 \geq 5 \\ & 3y_1 + \ \ y_2 + 4y_3 \geq 4 \\ & y_1 + 2y_2 + 2y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 0 \end{array} \tag{D}$$

It is extremely easy to find a feasible input $\mathbf{y}$: any one of $(3,0,0)$, $(0,4,0)$, or $(0,0,2)$ will work, for example, and there are infinitely many other possibilities. Any single vector like this is enough to show that this dual problem is not infeasible, i.e., $\min(D) < +\infty$. Therefore the extreme case of simultaneous infeasibility for both (P) and (D) is impossible, and we can rely on the extended equation

$$\max(P) = \min(D).$$

Knowing $\min(D) < +\infty$ implies $\max(P) < +\infty$, i.e., problem (P) cannot be unbounded.

5. The given dictionary is optimal for problem (P_0), so we would like to reuse it as much as possible. We can predict some of the troubles we will encounter by noticing that the unique optimum in (P_0) has $w_3 = \frac{1}{2}$: that is, the third inequality constraint is not active, but only because the upper limit of 7 is bigger than the left side by $\frac{1}{2}$. Reducing this upper limit to 6 will mean that the unique maximizer in (P_0) will not even be feasible in (P_1).

Changing the vector $\mathbf{b}$ in the original problem will influence only the column of constants in this dictionary. The constants we want come from setting all nonbasic variables to 0 in the original system

$$\begin{aligned}
2x_1 + x_2 + x_3 + w_1 \qquad\qquad &= 4 \\
3x_1 \qquad + 2x_3 \qquad + w_2 \qquad &= 5 \\
3x_1 + x_2 + 2x_3 \qquad\qquad + w_3 &= 6
\end{aligned}$$

to obtain the explicit form of $B\mathbf{x}_{\mathcal{B}} = \mathbf{b}$, namely,

$$\begin{aligned}
x_2 + x_3 \qquad &= 4 \\
2x_3 \qquad &= 5 \\
x_2 + 2x_3 + w_3 &= 6
\end{aligned}$$

The unique solution has $x_3 = 5/2$, so $x_2 = 3/2$, and then $w_3 = -1/2$.

The value of $f = 4x_1 + 2x_2 + 3x_3$ at this point is $21/2$.

So before the new constraint enters the picture, a dictionary for the modified problem differs from the given one by only one entry in the column of constants:

$$\begin{array}{rcl}
f &=& (21/2) - (3/2)x_1 - 2w_1 - (1/2)w_2 \\ \hline
x_2 &=& (3/2) - (1/2)x_1 - w_1 + (1/2)w_2 \\
x_3 &=& (5/2) - (3/2)x_1 \qquad - (1/2)w_2 \\
w_3 &=& (-1/2) + (1/2)x_1 + w_1 + (1/2)w_2
\end{array}$$

This dictionary is infeasible, as predicted.

To incorporate the new constraint $x_1 - x_2 - 2x_3 \leq -3$, we introduce a new slack variable w_4, declare it basic, and substitute from the dictionary above:

$$\begin{aligned}
w_4 &= -3 - x_1 + x_2 + 2x_3 \\
&= -3 - x_1 \\
&\quad (3/2) - (1/2)x_1 - w_1 + (1/2)w_2 \\
&\quad 5 - 3x_1 - w_2 \\
&= (9/2) - (7/2)x_1 - w_1 - (1/2)w_2
\end{aligned}$$

Adding this row to the dictionary we already have gives a dictionary for the modified problem (P_1):

$$\begin{array}{rcl}
f &=& (21/2) - (3/2)x_1 - 2w_1 - (1/2)w_2 \\ \hline
x_2 &=& (3/2) - (1/2)x_1 - w_1 + (1/2)w_2 \\
x_3 &=& (5/2) - (3/2)x_1 \qquad - (1/2)w_2 \\
w_3 &=& (-1/2) + (1/2)x_1 + w_1 + (1/2)w_2 \\
w_4 &=& (7/2) - (9/2)x_1 - w_1 - (1/2)w_2
\end{array}$$

This one is infeasible, too, but at least it's *dual-feasible*, so we can apply the Dual Simplex Method.

We want to pivot w_3 out of the basis. To choose an entering variable to replace it, we consider possible $t > 0$ in

$$f + tw_3 = \frac{21-t}{2} - \left(\frac{3-t}{2}\right)x_1 - (2-t)w_1 - \left(\frac{1-t}{2}\right)w_2.$$

The smallest $t > 0$ to create a zero coefficient here is $t = 1$, which chooses w_2 as the entering variable. The pivot equation is

$$w_2 = 1 - x_1 - 2w_1 + 2w_3,$$

and updating produces this dictionary:

$$\begin{array}{rcl} f & = & 10 - x_1 - w_1 - w_3 \\ \hline x_2 & = & 2 - x_1 - 2w_1 + w_3 \\ x_3 & = & 2 - x_1 + w_1 - w_3 \\ w_2 & = & 1 - x_1 - 2w_1 + 2w_3 \\ w_4 & = & 3 - 4x_1 \qquad\quad - w_3 \end{array}$$

This is optimal. The modified problem has $f_{\text{MAX}} = 10$, achieved uniquely by the input $\mathbf{x}^* = (0, 2, 2)$. (The corresponding slack values are $(w_1^*, w_2^*, w_3^*, w_4^*) = (0, 1, 0, 3)$.)

6. (a) Our problem is equivalent to $\max \left\{ \zeta = c^T x \,:\, Ax \le b,\ x \ge 0 \text{ in } \mathbb{R}^5 \right\}$, where

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad b = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix},$$

$$c^T = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

(b) When $x_2 = 0$ and $x_4 = 0$, the system $Ax = b$ reduces to

$$\left.\begin{array}{rcl} x_1 + x_3 & = & 3 \\ -x_1 & = & -1 \\ x_1 \qquad + x_5 & = & 4 \end{array}\right\}, \quad \text{which gives} \quad \begin{array}{l} x_1 = 1, \\ x_3 = 2, \\ x_5 = 3. \end{array}$$

This sets up all the basic information we need to start the RSM: $\mathcal{B} = \{1, 3, 5\}$, $\mathcal{N} = \{2, 4\}$,

$$B = \begin{array}{c} \begin{array}{ccc} x_1 & x_3 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}, \qquad N = \begin{array}{c} \begin{array}{cc} x_2 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{pmatrix} \end{array}, \qquad x_{\mathcal{B}}^* = \begin{array}{c} x_1 \\ x_3 \\ x_5 \end{array} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$c_{\mathcal{B}}^T = \begin{array}{c} \begin{array}{ccc} x_1 & x_3 & x_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \end{array}, \qquad c_{\mathcal{N}}^T = \begin{array}{c} \begin{array}{cc} x_2 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}, \qquad \zeta^* = 1.$$

(c) **Iteration 1.** (1) Select Entering Column. Start by solving for y^T in $y^T B = c_{\mathcal{B}}^T$, i.e.,

$$\left.\begin{array}{rcl} y_1 - y_2 + y_3 & = & 1 \\ y_1 \qquad\qquad & = & 0 \\ y_3 & = & 0 \end{array}\right\}, \quad \text{giving} \quad y^T = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}.$$

Then build the vector

$$\mathbf{z}_{\mathcal{N}}^T = y^T N - c_{\mathcal{N}} = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix} \begin{array}{c} \begin{array}{cc} x_2 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{cc} x_2 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} x_2 & x_4 \end{array} \\ \begin{pmatrix} -2 & -1 \end{pmatrix} \end{array}.$$

The smallest subscript (Bland's Rule) eligible to enter is 2.

(2) Select Leaving Column. Start by solving for d in $Bd = a^{(2)}$, i.e.,

$$\left.\begin{aligned} d_1 + d_2 \quad &= 1 \\ -\,d_1 \quad &= 1 \\ d_1 \quad + d_3 &= 2 \end{aligned}\right\} \quad \text{giving} \quad d = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

Use this to track the t-dependence in

$$x_{\mathcal{B}}(t) = x_{\mathcal{B}}^* - td = \begin{array}{c} x_1 \\ x_3 \\ x_5 \end{array} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - t \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{array}{c} x_1 \\ x_3 \\ x_5 \end{array} \begin{pmatrix} 1+t \\ 2-2t \\ 3-3t \end{pmatrix}. \qquad (**)$$

The smallest $t \ge 0$ to produce a zero entry is $t^* = 1$. It allows both x_3 and x_5 to leave the basis. We use Bland's Rule to pick x_3 to exit.

(3) Update. The new basis will be $\mathcal{B} = \{1, 2, 5\}$, giving $\mathcal{N} = \{3, 4\}$. We get $x_1^* = 2$ and $x_5^* = 0$ from $(**)$, and $x_2^* = t^* = 1$. Thus

$$B = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \end{array}, \qquad N = \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array}, \qquad x_{\mathcal{B}}^* = \begin{array}{c} x_1 \\ x_2 \\ x_5 \end{array} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$c_{\mathcal{B}}^T = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \end{array}, \qquad c_{\mathcal{N}}^T = \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} \end{array}, \qquad \zeta^* = 3.$$

Notice that this BFS is degenerate ($x_5^* = 0$).

Iteration 2. (1) Select Entering Column. Start by solving for y^T in $y^T B = c_{\mathcal{B}}^T$, i.e.,

$$\left.\begin{aligned} y_1 - y_2 + \ y_3 &= 1 \\ y_1 + y_2 + 2y_3 &= 1 \\ y_3 &= 0 \end{aligned}\right\} \quad \text{giving} \quad y^T = (\, 1 \quad 0 \quad 0 \,).$$

Then build and scan the row vector

$$\mathbf{z}_{\mathcal{N}}^T = y^T N - c_{\mathcal{N}} = (\, 1 \quad 0 \quad 0 \,) \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}.$$

This has no negative entries, so the current BFS is a maximizer. But the zero coefficient of nonbasic x_4 suggest that positive values of x_4 may be compatible with optimality. To probe the limits, imagine pivoting x_4 into the basis.

(2) Track the effect of choosing $x_4 = t$. Start by solving for d in $Bd = a^{(4)}$, i.e.,

$$\left.\begin{aligned} d_1 + \ d_2 \quad &= 0 \\ -\,d_1 + \ d_2 \quad &= 1 \\ d_1 + 2d_2 + d_3 &= 0 \end{aligned}\right\} \quad \text{giving} \quad d = \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

Then watch the basic variables respond when $x_4 = t$:

$$x_{\mathcal{B}}(t) = x_{\mathcal{B}}^* - td = \begin{array}{c} x_1 \\ x_2 \\ x_5 \end{array} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - t \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} = \begin{array}{c} x_1 \\ x_2 \\ x_5 \end{array} \begin{pmatrix} 2+t/2 \\ 1-t/2 \\ 0+t/2 \end{pmatrix}.$$

The first zero entry appears when $t = 2$. Thus the interval of feasible t-values is $0 \le t \le 2$.

Summary. We have $\zeta_{\max} = 3$, with the set of maximizers

$$(x_1, x_2) = (2 + t/2, 1 - t/2)\,, \qquad 0 \le t \le 2.$$

(This is the segment joining the points $(2,1)$ and $(3,0)$ in the (x_1, x_2)-plane.)

7. Name the slacks $x_5 = w_1$ and $x_6 = w_2$ so that subscripts between 1 and 6 can be used to identify variables of either type. The problem becomes $\max\left\{c^T x \,:\, Ax = b,\ x \ge 0 \text{ in } \mathbb{R}^6\right\}$, with

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 3 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 & 1 \end{pmatrix} \end{array}, \qquad b = \begin{pmatrix} 5 \\ 3 \end{pmatrix},$$

$$c^T = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 6 & 8 & 5 & 9 & 0 & 0 \end{pmatrix} \end{array}.$$

A feasible basis is obvious: $\mathcal{B} = \{5, 6\}$. It gives $\mathcal{N} = \{1, 2, 3, 4\}$ and the partitioning scheme

$$B = \begin{array}{c} \begin{array}{cc} x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}, \qquad N = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 1 & 2 \end{pmatrix} \end{array}, \qquad x_{\mathcal{B}}^* = \begin{array}{c} x_5 \\ x_6 \end{array} \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$c_{\mathcal{B}}^T = \begin{array}{c} \begin{array}{cc} x_5 & x_6 \end{array} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} \end{array}, \qquad c_{\mathcal{N}}^T = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 6 & 8 & 5 & 9 \end{pmatrix} \end{array}, \qquad \zeta^* = 0.$$

Iteration 1. (1) Select Entering Variable. First, solve for y in $y^T B = c_{\mathcal{B}}^T$. This is easy because $B = I$ and $c_{\mathcal{B}} = 0$: we get $y = 0$. Second, form the row vector

$$\mathbf{z}_{\mathcal{N}}^T = y^T N - c_{\mathcal{N}}^T = (0 \quad 0) \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 1 & 2 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 6 & 8 & 5 & 9 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} -6 & -8 & -5 & -9 \end{pmatrix} \end{array}.$$

Scan this vector for the negative entry of greatest magnitude. Deduce that x_4 must enter the basis.

(2) Select Leaving Variable. First, solve for d in $Bd = a^{(4)}$. This is easy because $B = I$: it gives $d = (3, 2)$. Second, study the t-dependence of

$$x_{\mathcal{B}}(t) = x_{\mathcal{B}}^* - td = \begin{array}{c} x_5 \\ x_6 \end{array} \begin{pmatrix} 5 \\ 3 \end{pmatrix} - t \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{array}{c} x_5 \\ x_6 \end{array} \begin{pmatrix} 5 - 3t \\ 3 - 2t \end{pmatrix}. \qquad (**)$$

The smallest $t \ge 0$ that produces a zero component here is $t^* = 3/2$. It selects x_6 to leave the basis.

(3) Update. The new basis set will be $\mathcal{B} = \{4, 5\}$, so $\mathcal{N} = \{1, 2, 3, 6\}$. We will get $x_5^* = 1/2$ from $(**)$ and $x_4^* = t^* = 3/2$. The new situation is this:

$$B = \begin{array}{c} \begin{array}{cc} x_4 & x_5 \end{array} \\ \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \end{array}, \qquad N = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_6 \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \end{pmatrix} \end{array}, \qquad x_{\mathcal{B}}^* = \begin{array}{c} x_4 \\ x_5 \end{array} \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

$$c_{\mathcal{B}}^T = \begin{array}{c} \begin{array}{cc} x_4 & x_5 \end{array} \\ \begin{pmatrix} 9 & 0 \end{pmatrix} \end{array}, \qquad c_{\mathcal{N}}^T = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_6 \end{array} \\ \begin{pmatrix} 6 & 8 & 5 & 0 \end{pmatrix} \end{array}, \qquad \zeta^* = 27/2.$$

Iteration 2. (1) Select Entering Variable. First, solve for y in $y^T B = c_{\mathcal{B}}^T$, i.e.,

$$\left.\begin{aligned} 3y_1 + 2y_2 &= 9 \\ y_1 \qquad &= 0 \end{aligned}\right\} \quad \text{giving} \quad y^T = (\,0 \quad 9/2\,).$$

Second, build the row vector

$$\mathbf{z}_{\mathcal{N}}^T = y^T N - c_{\mathcal{N}}^T = (\,0 \quad 9/2\,) \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_6 \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_6 \end{array} \\ (\,6 \quad 8 \quad 5 \quad 0\,) \end{array} = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_6 \end{array} \\ \left(-\frac{3}{2} \quad \frac{11}{2} \quad -\frac{1}{2} \quad \frac{9}{2}\right) \end{array}.$$

Scan this vector for the most negative entry. Deduce that x_1 must enter the basis.

(2) Select Leaving Variable. First, solve for d in $Bd = a^{(1)}$, i.e.,

$$\left.\begin{aligned} 3d_1 + d_2 &= 2 \\ 2d_1 \qquad &= 1 \end{aligned}\right\} \quad \text{giving} \quad d = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Second, study the t-dependence of

$$x_{\mathcal{B}}(t) = x_{\mathcal{B}}^* - td = \begin{array}{c} x_4 \\ x_5 \end{array} \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} - t \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{array}{c} x_4 \\ x_5 \end{array} \begin{pmatrix} (3/2) - t/2 \\ (1/2) - t/2 \end{pmatrix}. \qquad (**)$$

The smallest $t \geq 0$ that produces a zero component here is $t^* = 1$. It identifies x_5 as the leaving variable.

(3) Update. The new basis set will be $\mathcal{B} = \{1, 4\}$, so $\mathcal{N} = \{2, 3, 5, 6\}$. We will get $x_4^* = 1$ from $(**)$ and $x_1^* = t^* = 1$. The new situation is this:

$$B = \begin{array}{c} \begin{array}{cc} x_1 & x_4 \end{array} \\ \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \end{array}, \qquad N = \begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \end{array}, \qquad x_{\mathcal{B}}^* = \begin{array}{c} x_1 \\ x_4 \end{array} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$c_{\mathcal{B}}^T = \begin{array}{c} \begin{array}{cc} x_1 & x_4 \end{array} \\ (\,6 \quad 9\,) \end{array}, \qquad c_{\mathcal{N}}^T = \begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_5 & x_6 \end{array} \\ (\,8 \quad 5 \quad 0 \quad 0\,) \end{array}, \qquad \zeta^* = 15.$$

Iteration 3. (1) Select Entering Variable. First, solve for y in $y^T B = c_{\mathcal{B}}^T$, i.e.,

$$\left.\begin{aligned} 2y_1 + \ \ y_2 &= 6 \\ 3y_1 + 2y_2 &= 9 \end{aligned}\right\} \quad \text{giving} \quad y^T = (\,3 \quad 0\,).$$

Second, build the row vector

$$\mathbf{z}_{\mathcal{N}}^T = y^T N - c_{\mathcal{N}}^T = (\,3 \quad 0\,) \begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \end{array} - \begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_5 & x_6 \end{array} \\ (\,8 \quad 5 \quad 0 \quad 0\,) \end{array} = \begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_5 & x_6 \end{array} \\ (\,-5 \quad -2 \quad 3 \quad 0\,) \end{array}.$$

The most negative entry selects x_2 as the entering variable.

(2) Select Leaving Variable. First, solve for d in $Bd = a^{(2)}$, i.e.,

$$\left.\begin{aligned} 2d_1 + 3d_2 &= 1 \\ d_1 + 2d_2 &= 3 \end{aligned}\right\} \quad \text{giving} \quad d = \begin{pmatrix} -7 \\ 5 \end{pmatrix}.$$

Second, study the t-dependence of

$$x_{\mathcal{B}}(t) = x_{\mathcal{B}}^* - td = \begin{array}{c} x_1 \\ x_4 \end{array} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - t \begin{pmatrix} -7 \\ 5 \end{pmatrix} = \begin{array}{c} x_1 \\ x_4 \end{array} \begin{pmatrix} 1 + 7t \\ 1 - 5t \end{pmatrix}.$$

The smallest $t \geq 0$ that produces a zero component here is $t^* = 1/5$. It identifies x_4 as the leaving variable.

(3) Update. The new basis set will be $\mathcal{B} = \{1, 2\}$, so $\mathcal{N} = \{3, 4, 5, 6\}$. We will get $x_1^* = 12/5$ from (**) and $x_2^* = t^* = 1/5$. The new situation is this:

$$B = \begin{array}{cc} x_1 & x_2 \\ \left(\begin{array}{c} 2 \\ 1 \end{array}\right. & \left.\begin{array}{c} 1 \\ 3 \end{array}\right) \end{array}, \qquad N = \begin{array}{cccc} x_3 & x_4 & x_5 & x_6 \\ \left(\begin{array}{c} 1 \\ 1 \end{array}\right. & \begin{array}{c} 3 \\ 2 \end{array} & \begin{array}{c} 1 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 1 \end{array}\right) \end{array}, \qquad x_{\mathcal{B}}^* = \begin{array}{c} x_1 \\ x_2 \end{array} \begin{pmatrix} 12/5 \\ 1/5 \end{pmatrix}$$

$$c_{\mathcal{B}}^T = \begin{array}{cc} x_1 & x_2 \\ (\,6 & 8\,) \end{array}, \qquad c_{\mathcal{N}}^T = \begin{array}{cccc} x_3 & x_4 & x_5 & x_6 \\ (\,5 & 9 & 0 & 0\,) \end{array}, \qquad \zeta^* = 16.$$

Iteration 4. (1) Select Entering Variable. First, solve for y in $y^T B = c_{\mathcal{B}}^T$, i.e.,

$$\left.\begin{aligned} 2y_1 + y_2 &= 6 \\ y_1 + 3y_2 &= 8 \end{aligned}\right\} \quad \text{giving} \quad y^T = (\,2 \quad 2\,).$$

Second, build the row vector

$$\mathbf{z}_{\mathcal{N}}^T = y^T N - c_{\mathcal{N}}^T = (\,2 \quad 2\,) \begin{array}{cccc} x_3 & x_4 & x_5 & x_6 \\ \left(\begin{array}{c} 1 \\ 1 \end{array}\right. & \begin{array}{c} 3 \\ 2 \end{array} & \begin{array}{c} 1 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 1 \end{array}\right) \end{array} - \begin{array}{cccc} x_3 & x_4 & x_5 & x_6 \\ (\,5 & 9 & 0 & 0\,) \end{array} = \begin{array}{cccc} x_3 & x_4 & x_5 & x_6 \\ (\,-1 & 1 & 2 & 2\,) \end{array}.$$

The largest positive entry identifies x_3 as the entering variable.

(2) Select Leaving Variable. First, solve for d in $Bd = a^{(3)}$, i.e.,

$$\left.\begin{aligned} 2d_1 + d_2 &= 1 \\ d_1 + 3d_2 &= 1 \end{aligned}\right\} \quad \text{giving} \quad d = \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix}.$$

Then watch the t-dependence of

$$x_{\mathcal{B}}(t) = x_{\mathcal{B}}^* - td = \begin{array}{c} x_1 \\ x_2 \end{array} \begin{pmatrix} 12/5 \\ 1/5 \end{pmatrix} - t \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix} = \begin{array}{c} x_1 \\ x_2 \end{array} \begin{pmatrix} 12/5 - 2t/5 \\ 1/5 - t/5 \end{pmatrix}. \tag{**}$$

The smallest $t \geq 0$ that produces a zero is $t^* = 1$. It identifies x_2 as the leaving variable.

(3) Update. The new basis will be $\mathcal{B} = \{1, 3\}$, so $\mathcal{N} = \{2, 4, 5, 6\}$. We get $x_1^* = 2$ from (**) and $x_2^* = t^* = 1$. The new situation is as follows.

$$B = \begin{array}{cc} x_1 & x_3 \\ \left(\begin{array}{c} 2 \\ 1 \end{array}\right. & \left.\begin{array}{c} 1 \\ 1 \end{array}\right) \end{array}, \qquad N = \begin{array}{cccc} x_2 & x_4 & x_5 & x_6 \\ \left(\begin{array}{c} 1 \\ 3 \end{array}\right. & \begin{array}{c} 3 \\ 2 \end{array} & \begin{array}{c} 1 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 1 \end{array}\right) \end{array}, \qquad x_{\mathcal{B}}^* = \begin{array}{c} x_1 \\ x_3 \end{array} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$c_{\mathcal{B}}^T = \begin{array}{cc} x_1 & x_3 \\ (\,6 & 5\,) \end{array}, \qquad c_{\mathcal{N}}^T = \begin{array}{cccc} x_2 & x_4 & x_5 & x_6 \\ (\,8 & 9 & 0 & 0\,) \end{array}, \qquad \zeta^* = 17.$$

Iteration 5. (1) Select Entering Variable. First, solve for y in $y^T B = c_{\mathcal{B}}^T$, i.e.,

$$\left.\begin{aligned} 2y_1 + y_2 &= 6 \\ y_1 + y_2 &= 5 \end{aligned}\right\} \quad \text{giving} \quad y^T = (\,1 \quad 4\,).$$

Second, build the row vector

$$\mathbf{z}_{\mathcal{N}}^T = y^T N - c_{\mathcal{N}}^T = (\,1 \quad 4\,) \begin{array}{cccc} x_2 & x_4 & x_5 & x_6 \\ \left(\begin{array}{c} 1 \\ 3 \end{array}\right. & \begin{array}{c} 3 \\ 2 \end{array} & \begin{array}{c} 1 \\ 0 \end{array} & \left.\begin{array}{c} 0 \\ 1 \end{array}\right) \end{array} - \begin{array}{cccc} x_2 & x_4 & x_5 & x_6 \\ (\,8 & 9 & 0 & 0\,) \end{array} = \begin{array}{cccc} x_2 & x_4 & x_5 & x_6 \\ (\,5 & 2 & 1 & 4\,) \end{array}.$$

This vector has only positive components, so the current BFS is the unique maximizer.

Summary. The maximizer in this problem is unique:

$$(x_1^*, x_2^*, x_3^*, x_4^*) = (2, 0, 1, 0); \qquad \zeta_{\max} = 17.$$