Math 340(921) Problem Set 5

Due in class on Friday 14 June 2013

1. Show that the following problem is dual to itself:

$$
\begin{array}{lrl}
\text{maximize} & \zeta = & x_1 - 2x_2 - 3x_3 \\
\text{subject to} & & \phantom{x_1 -{}} x_2 + 2x_3 \geq 1 \\
 & & x_1 \phantom{{}- 2x_2} + 3x_3 \leq 2 \\
 & & 2x_1 - 3x_2 \phantom{{}+3x_3} = 3 \\
 & & x_1, x_2 \geq 0,\ x_3 \text{ unrestricted}
\end{array}
$$

2. Find all maximizers in this LP by following the method detailed below:

$$
\begin{array}{lrl}
\text{maximize } f = & 2x_1 - 6x_2 & \\
\text{subject to} & -x_1 - x_2 - x_3 \leq & -2 \\
 & 2x_1 - x_2 + x_3 \leq & 1 \\
 & x_1, x_2, x_3 \geq 0. &
\end{array}
$$

We'll need a feasible dictionary before we can start Phase II. To get one (without using the Phase I introduced earlier), replace the the primal objective f with

$$\widetilde{f} = -x_1 - x_2 - x_3.$$

(This $\widetilde{f}$ is obviously dual-feasible. Any dual-feasible choice could be used, but marking will be easier if all students make the same choice!) Apply Dual Simplex pivots to the primal dictionary based on $\widetilde{f}$ to get a feasible dictionary, then put f back in place and finish as usual.

(*On your own, contemplate the advantages and disadvantages of this approach relative to the Phase I method presented earlier in the course.*)

3. Consider the following standard-form problem, stated here with its dual:

$$
\begin{array}{lrl}
\text{Maximize } f = 4x_1 + 5x_2 & & \\
\text{subject to} & x_1 + x_2 \leq & 5 \\
 & x_1 + 2x_2 \leq & 8 \\
 & 2x_1 + x_2 \leq & 9 \\
 & x_1, x_2 \geq & 0
\end{array}
\qquad\Bigg|\qquad
\begin{array}{lrl}
\text{Minimize } g = 5y_1 + 8y_2 + 9y_3 & & \\
\text{subject to} & y_1 + y_2 + 2y_3 \geq & 4 \\
 & y_1 + 2x_2 + y_3 \geq & 5 \\
 & y_1, y_2, y_3 \geq & 0
\end{array}
$$

Introducing primal slack variables w_i and dual surplus variables z_j leads to a primal-dual pair of dictionaries:

$$
\begin{array}{rcl}
f & = & 4x_1 + 5x_2 \\
\hline
w_1 & = & 5 - x_1 - x_2 \\
w_2 & = & 8 - x_1 - 2x_2 \\
w_3 & = & 9 - 2x_1 - x_2
\end{array}
\qquad\Bigg|\qquad
\begin{array}{rcl}
-g & = & -5y_1 - 8y_2 - 9y_3 \\
\hline
z_1 & = & -4 + y_1 + y_2 + 2y_3 \\
z_2 & = & -5 + y_1 + 2y_2 + y_3
\end{array}
$$

Complementary slackness couples the primal variables $(x_1, x_2; w_1, w_2, w_3)$ with the dual variables $(z_1, z_2; y_1, y_2, y_3)$. Since w_1, w_2, w_3 are basic in the primal, the corresponding variables y_1, y_2, y_3 are nonbasic in the dual.

Use two columns of equal width, side by side, to present your results for parts (a), (b), and (d).

(a) In the left column, copy the dictionary above and then solve the primal problem using the simplex method. (It only takes 2 pivots.) Clearly state which variables pivot into and out of the basis at each step. For each dictionary, write the vector of primal variables in order and circle the basic ones.

More . . .

(b) In the right column, for each primal dictionary, write the vector of dual variables and circle the basic ones. State which ones are entering and leaving the dual basis. Given these selections, calculate the new dual dictionary by pivoting in the usual way.

(c) Study the relationship between the primal dictionaries on the left and the dual dictionaries on the right. Then write a clear statement about how the columns in the primal dictionary correspond to the rows in its parallel dual dictionary.

(d) Apply your statement in (c) to recover the primal dictionary corresponding to the dual dictionary given in the right column below.

$$\begin{array}{rcl} -g &=& -18 - (9/2)z_1 - (1/2)y_1 - (7/2)y_2 \\ \hline z_2 &=& -3 + (1/2)z_1 + (1/2)y_1 + (3/2)y_2 \\ y_3 &=& 2 + (1/2)z_1 - (1/2)y_1 - (1/2)y_2 \end{array}$$

4. The standard-form problem (2.1) on page 13 of the textbook has its optimal dictionary shown in line (2.10) on page 17.

(a) Use the Dual Simplex Method (dictionary form) to solve the related problem in which the second constraint is replaced by

$$4x_1 + x_2 + 2x_3 \leq 4.$$

(This is the only change.)

(b) Suppose, instead, we replace the second constraint in problem (2.1) with

$$4x_1 + x_2 + 2x_3 \leq -\theta,$$

where θ is a positive constant.

(i) Give a simple direct explanation for why the new problem is infeasible.

(ii) Explain how infeasibility would be revealed if we applied the dual simplex method to this modified problem. (When infeasibility is not obvious, and something strange happens in the dual simplex method, it's good to recognize the reason.)

(c) Do there exist real constants α, β, δ for which the following problem is unbounded? If so, find constants that do the job; if not, clearly explain why not.

$$\begin{array}{rl} \text{Maximize} & 5x_1 + 4x_2 + 3x_3 \\ \text{Subject to} & 2x_1 + 3x_2 + x_3 \leq \alpha \\ & 4x_1 + x_2 + 2x_3 \leq \beta \\ & 3x_1 + 4x_2 + 2x_3 \leq \delta \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

5. A standard-form problem and one of its dictionaries are given below. Slack variables are named w_j.

$$(P_0) \qquad \begin{array}{rl} \text{Maximize } f = & 4x_1 + 2x_2 + 3x_3 \\ \text{subject to} & 2x_1 + x_2 + x_3 \leq 4 \\ & 3x_1 + 2x_3 \leq 5 \\ & 3x_1 + x_2 + 2x_3 \leq 7 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$$\begin{array}{rcl} f &=& (21/2) - (3/2)x_1 - 2w_1 - (1/2)w_2 \\ \hline x_2 &=& (3/2) - (1/2)x_1 - w_1 + (1/2)w_2 \\ x_3 &=& (5/2) - (3/2)x_1 - (1/2)w_2 \\ w_3 &=& (1/2) + (1/2)x_1 + w_1 + (1/2)w_2 \end{array}$$

Here is related problem. It involves the same variables and objectives as the problem above, but a new constraint appears and the resource levels are a little different.

$$
(P_1) \qquad \begin{array}{ll}
\text{Maximize } \widetilde{f} = 4x_1 + 2x_2 + 3x_3 & \\
\text{subject to} \quad 2x_1 + x_2 + x_3 \leq & 4 \\
3x_1 \qquad\; + 2x_3 \leq & 5 \\
3x_1 + x_2 + 2x_3 \leq & 6 \\
x_1 - x_2 - 2x_3 \leq & -3 \\
x_1, x_2, x_3 \geq 0 &
\end{array}
$$

Find an optimal dictionary for problem (P_1), and identify all maximizing vectors $\mathbf{x}$ in $\mathbb{R}^3$.

6. It is rather obvious (by sketch or by algebra) that $x^* = (2, 1)$ solves the following problem:

$$
\begin{array}{lrl}
\text{maximize } \zeta = & x_1 + x_2 & \\
\text{subject to} & x_1 + x_2 \leq & 3 \\
& -x_1 + x_2 \leq & -1 \\
& x_1 + 2x_2 \leq & 4 \\
& x_1, x_2 \geq 0 &
\end{array}
$$

The point of this exercise is to watch the Revised Simplex Method track down the maximizer.

(a) Introduce slack variables $(x_3, x_4, x_5) = (w_1, w_2, w_3)$ to rewrite this problem in a form to which the Revised Simplex method (RSM) applies.

(b) Show that the basis $\mathcal{B} = \{1, 3, 5\}$ is feasible by solving a linear system to find $x^*_{\mathcal{B}}$ for this $\mathcal{B}$.

(c) Starting from the basis in part (b), use Bland's Rules to select pivots in the RSM. Repeat until you can find all maximizing vectors $x^* \in \mathbb{R}^5$.

7. Use the Revised Simplex Method, by hand, to solve the following LP:

$$
\begin{array}{ll}
\text{maximize } \zeta = & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\
\text{subject to} & 2x_1 + x_2 + x_3 + 3x_4 \leq 5 \\
& x_1 + 3x_2 + x_3 + 2x_4 \leq 3 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Choose pivots using Anstee's Rules.