M340(921) Solutions—Problem Set 4

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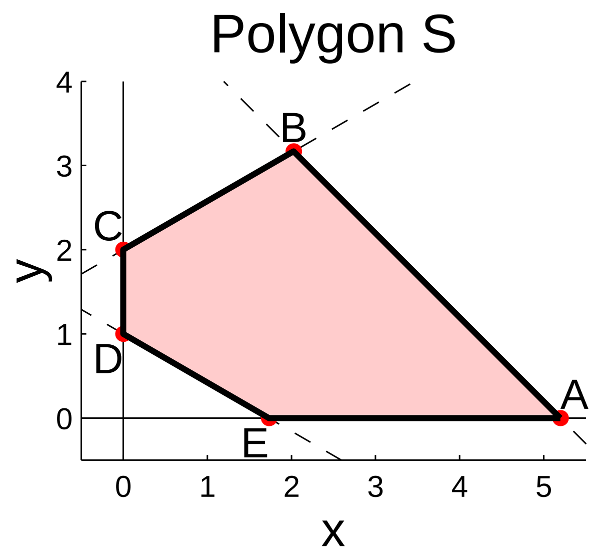
1. (a) Here is a sketch of the polygonal region S:



(b) The vertex coordinates of points in S, as labelled in the figure, are

$$A(3\sqrt{3},0), \quad B\left(\frac{9-2\sqrt{3}}{1+\sqrt{3}}, \frac{5\sqrt{3}}{1+\sqrt{3}}\right), \quad C(0,2), \quad D(0,1), \quad E(\sqrt{3},0).$$

(c) A line that makes an angle θ with the positive x-axis has slope $m = \tan\theta$. The given lines have slopes

$$\frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}, \quad -\frac{1}{\sqrt{3}} = \tan\left(-\frac{\pi}{6}\right), \quad \text{and} \quad -1 = \tan\left(-\frac{\pi}{4}\right).$$

This makes the interior angles at the points on an axis easy to find:

$$\angle A = \frac{\pi}{4}, \quad \angle C = \frac{2\pi}{3}, \quad \angle D = \frac{2\pi}{3}, \quad \angle E = \frac{5\pi}{6}.$$

The interior angles of every 5-sided polygon sum to 3π. Therefore $\angle B = \dfrac{7\pi}{12}$.

(In degrees, $(\angle A, \angle B, \angle C, \angle D, \angle E) = (45, 105, 120, 120, 150)$. But *radians* were specified.)

(d) To maximize $f(x,y) = x - y$, we want the largest possible value for x and the smallest possible value for y. Clearly the point $A(3\sqrt{3},0)$ has these properties, and the labels shown above have been chosen to match.

(e) For each nonzero vector $\mathbf{c} = (c_1, c_2)$, the points in S lie on the boundary. They are precisely the points where the vector $\mathbf{c} = (c_1, c_2)$ is normal to a line that contains no interior points of S. To summarize the results in written form, associate the angle θ with $\mathbf{c}$ exactly when

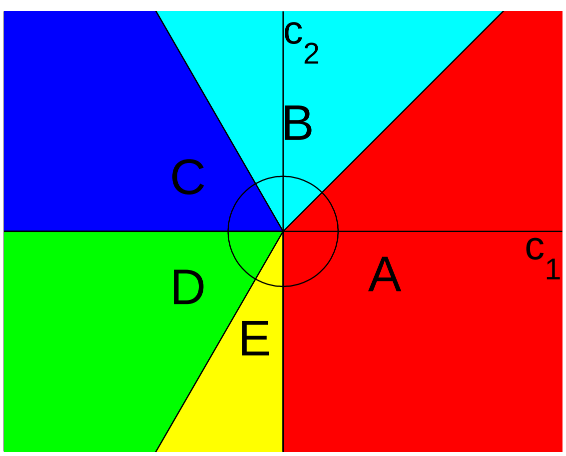
$$\mathbf{c} = |\mathbf{c}|\,(\cos\theta, \sin\theta)\,.$$

Except for five rays discussed below, there is a unique maximizing point in S, and it is one of the vertices labelled above:

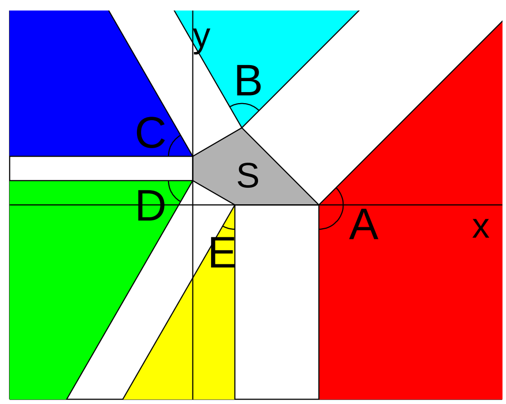
$$-\frac{\pi}{2} < \theta < \frac{\pi}{4} \implies \text{unique maximizer is } A(3\sqrt{3},0); \quad Z(c) = 3\sqrt{3}\,c_1$$
$$\frac{\pi}{4} < \theta < \frac{2\pi}{3} \implies \text{unique maximizer is } B; \quad Z(c) = \frac{(9-2\sqrt{3})c_1 + 5\sqrt{3}\,c_2}{1+\sqrt{3}}$$
$$\frac{2\pi}{3} < \theta < \pi \implies \text{unique maximizer is } C(0,2); \quad Z(c) = 2c_2$$
$$\pi < \theta < \frac{4\pi}{3} \implies \text{unique maximizer is } D(0,1); \quad Z(c) = c_2$$
$$\frac{4\pi}{3} < \theta < \frac{3\pi}{2} \implies \text{unique maximizer is } E(\sqrt{3},0); \quad Z(c) = \sqrt{3}\,c_1$$

On the rays between the wedges just given, two adjacent vertices both achieve the maximum and all points on the segment between them do, too. For example, the set of maximizers when $\theta = \pi/4$ is the segment $\overline{AB}$, including both endpoints. The first figure below shows the desired map.

Map of Maximizing Vertices



Feasible Set with Objective Map

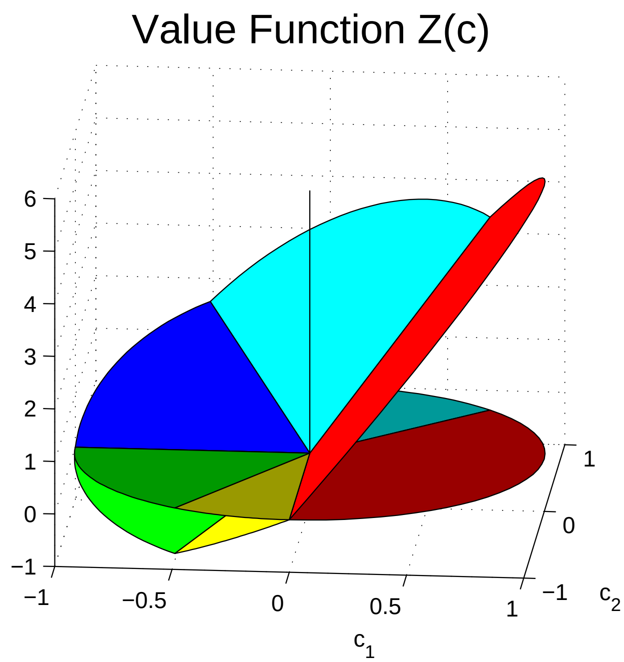


Discussion (not required): The map has a nice geometrical relationship to the original set S. The second figure illustrates it. When each coloured wedge from the map is positioned at its corresponding vertex, the faces of the map are orthogonal to the faces of S that meet at that vertex. (Objective vectors that select a whole face of the constraint set are not shown here. Please think about how to add them.)

(f) Most of the requested Z-values are embedded in the solution to part (e). For completeness, we show the borderline cases here.

$$\theta = \frac{\pi}{4}: \quad c_1 = c_2 > 0, \quad Z(c) = 3\sqrt{3}\,c_1 = 3\sqrt{3}\,c_2$$
$$\theta = \frac{2\pi}{3}: \quad c_2 = -\sqrt{3}c_1 > 0, \quad Z(c) = -2\,c_1 = \frac{2\sqrt{3}}{3}\,c_2$$
$$\theta = -\pi: \quad c_1 < 0,\ c_2 = 0, \quad Z(c) = 0$$
$$\theta = 4\pi/3: \quad c_2 = \sqrt{3}\,c_1 < 0, \quad Z(c) = \sqrt{3}\,c_1 = c_2$$
$$\theta = -\pi/2: \quad c_1 = 0,\ c_2 < 0, \quad Z(c) = 0$$

Remarks (Not Required from Students): Here is a sketch showing part of the graph of function Z. This is the surface $z = Z(c_1, c_2)$ above the (c_1, c_2)-plane; a disk in the plane is shown with the same colour codes used in earlier parts.



2. Here we copy the given problem and then write its dual beside it:

$$\begin{array}{ll} \text{maximize} & \zeta = -x_1 - 2x_2 \\ \text{subject to} & -2x_1 + 7x_2 \le 6 \\ & -3x_1 + x_2 \le -1 \\ & 9x_1 - 4x_2 \le 6 \\ & x_1 - x_2 \le 1 \\ & 7x_1 - 3x_2 \le 6 \\ & -5x_1 + 2x_2 \le -3 \\ & x_1, x_2 \ge 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & \xi = 6y_1 - y_2 + 6y_3 + y_4 + 6y_5 - 3y_6 \\ \text{subject to} & -2y_1 - 3y_2 + 9y_3 + y_4 + 7y_5 - 5y_6 \ge -1 \\ & 7y_1 + y_2 - 4y_3 - y_4 - 3y_5 + 2y_6 \ge -2 \end{array}$$

The dual problem has a feasible origin. Transforming to standard maximization form and introducing slack variables z_1, z_2, we arrive at the dictionary

$$\begin{array}{rl} -\xi = & -6y_1 + y_2 - 6y_3 - y_4 - 6y_5 + 3y_6 \\ \hline z_1 = & 1 - 2y_1 - 3y_2 + 9y_3 + y_4 + 7y_5 - 5y_6 \\ z_2 = & 2 + 7y_1 + y_2 - 4y_3 - y_4 - 3y_5 + 2y_6 \end{array}$$

Only y_2 and y_6 are eligible to enter. Choosing with the largest-coefficient rule, we let y_6 enter the basis. This forces z_1 to leave, and leads to the dictionary written below. (The pivot equation appears under the objective row.)

$$
\begin{array}{r}
-\xi = 0.6 - 7.2y_1 - 0.8y_2 - 0.6y_3 - 0.4y_4 - 1.8y_5 - 0.6z_1 \\
\hline
y_6 = 0.2 - 0.4y_1 - 0.6y_2 + 1.8y_3 + 0.2y_4 + 1.4y_5 - 0.2z_1 \\
z_2 = 2.4 + 6.2y_1 - 0.2y_2 - 0.4y_3 - 0.6y_4 - 0.2y_5 - 0.4z_1
\end{array}
$$

This is an optimal dictionary, with

$$(-\xi)_{\max} = 0.6 \quad \text{at} \quad (y_1^*, y_2^*, y_3^*, y_4^*, y_5^*, y_6^*) = (0,0,0,0,0,0.2); (z_1^*, z_2^*) = (0, 2.4). \qquad (**)$$

It follows that $\zeta_{\max} = \xi_{\min} = -0.6$. The point that attains the primal maximum comes from negating the coefficients of the dual slacks in the optimal dictionary just written. Thus we have the primal solution

$$\zeta_{\max} = -0.6 \quad \text{at} \quad (x_1^*, x_2^*) = (0.6, 0). \qquad (*)$$

It's easy to double-check that this is correct: the vector y^* in line $(**)$ is dual-feasible, the vector x^* in $(*)$ is primal-feasible, and $\zeta(x^*) = \xi(y^*)$. These three conditions provide certain confirmation that x^* and y^* are simultaneous optimizers in their respective problems.

3. (a) The lower rows of the given dictionary encode the problem's constraints:

$$
\begin{aligned}
x_1 &= 1 + x_2 - 2x_4 + x_6 \\
x_3 &= 3 - 4x_2 + 3x_4 - 2x_6 \\
x_5 &= 2 + 3x_2 + 2x_4
\end{aligned}
$$

In the initial dictionary, the slack variables are basic. Work backward toward this. Pivot x_6 into the basis and x_1 out:

$$
\begin{aligned}
x_6 &= -1 - x_2 + 2x_4 + x_1 \\
x_3 &= 5 - 2x_2 - x_4 - 2x_1 \\
x_5 &= 2 + 3x_2 + 2x_4
\end{aligned}
$$

Next, pivot x_4 into the basis and x_3 out. Re-order the equations to get

$$
\begin{aligned}
x_4 &= 5 - 2x_1 - 2x_2 - x_3 \\
x_5 &= 12 - 4x_1 - x_2 - 2x_3 \\
x_6 &= 9 - 3x_1 - 5x_2 - 2x_3
\end{aligned}
$$

Using these equations, we can express the objective row as

$$\zeta = 15 - x_2 - 3x_4 = 15 - x_2 - 3(5 - 2x_1 - 2x_2 - x_3) = 6x_1 + 5x_2 + 3x_3.$$

Here is the original standard-form LP:

$$
(P) \quad \left[\begin{array}{ll}
\text{Maximize} & \zeta = 6x_1 + 5x_2 + 3x_3 \\
\text{subject to} & 2x_1 + 2x_2 + x_3 \le 5 \\
& 4x_1 + x_2 + 2x_3 \le 12 \\
& 3x_1 + 5x_2 + 2x_3 \le 9 \\
& x_1, x_2, x_3 \ge 0
\end{array}\right.
$$

(b) The given dictionary is optimal, so $\zeta_{\max} = 15$. To achieve this value requires $x_2 = 0$ and $x_4 = 0$, but allows some positive values for the nonbasic variable x_6. The dictionary rows give the parametric list of maximizers

$$\mathbf{x} = (1 + x_6, 0, 3 - 2x_6, 0, 2, x_6), \qquad 0 \le x_6 \le \frac{3}{2}.$$

Here the permitted interval of values for x_6 is the largest one compatible with the componentwise constraint $\mathbf{x} \ge \mathbf{0}$.

(c) The given dictionary is optimal, so the dual minimizer can be read out of the objective row: $\mathbf{y}^* = (3, 0, 0)$. In particular, if Z denotes the maximum value in the problem as a function of the constraint components (b_1, b_2, b_3), we have $Z = 15$ right now, and

$$\frac{\partial Z}{\partial b_1} = y_1^* = 3, \quad \frac{\partial Z}{\partial b_2} = y_2^* = 0, \quad \frac{\partial Z}{\partial b_3} = y_3^* = 0.$$

The only parameter with a perceptible effect on Z is b_1. To get $\Delta Z = +2$, we need

$$+2 = \Delta Z \approx \left(\frac{\partial Z}{\partial b_1}\right) \Delta b_1 = 3\Delta b_1.$$

That is, $\Delta b_1 \approx 2/3$. We should change the right side of the first constraint from 5 to $5 + 2/3 = 17/3$. (This change actually gives $Z = 17$ exactly, but detailed confirmation was not requested.)

(d) The dual for problem (P) (see part (a) above) is

$$(D) \quad \left[\begin{array}{lrl} \text{Minimize} & \xi = & 5y_1 + 12y_2 + 9y_3 \\ \text{subject to} & & 2y_1 + 4y_2 + 3y_3 \ge 6 \\ & & 2y_1 + y_2 + 5y_3 \ge 5 \\ & & y_1 + 2y_2 + 2y_3 \ge 3 \\ & & y_1, y_2, y_3 \ge 0 \end{array}\right.$$

The dual dictionary corresponding to the optimal primal dictionary given in the setup can be found by exploiting the negative-transpose property (or by pivoting):

$$\begin{array}{lcl} -\xi = & -15 - & z_1 - 3z_3 - 2y_2 \\ \hline z_2 = & 1 - & z_1 + 4z_3 - 3y_2 \\ y_1 = & 3 + & 2z_1 - 3z_3 - 2y_2 \\ y_3 = & 0 - & z_1 + 2z_3 \end{array}$$

It reveals that $(-\xi)_{\max} = -15$, attained only when $z_1 = 0$, $z_3 = 0$, and $y_2 = 0$. This produces the unique dual minimizer $y^* = (3, 0, 0)$ with dual slacks $z^* = (0, 1, 0)$.

(d′) [Alternative] **Every** dual minimizer must collaborate with **every** primal maximizer to satisfy the complementary slackness conditions. Let's consider the primal maximizer obtained when $x_6^* = 1$, namely,

$$x^* = (x_1^*, x_2^*, x_3^*) = (2, 0, 1); \qquad w^* = (x_4^*, x_5^*, x_6^*) = (0, 2, 1).$$

(i) Clearly both $x^* \ge 0$ and $w^* \ge 0$.

(ii) Since $w_2^* > 0$ and $w_3^* > 0$, any dual minimizer must obey $y_2^* = 0$ and $y_3^* = 0$.

(iii) Since $x_1^* > 0$ and $x_3^* > 0$, the slacks for a true dual minimizer must obey

$$0 = z_1^* = 2y_1^* + 4y_2^* + 3y_3^* - 6$$

$$0 = z_3^* = y_1^* + 2y_2^* + 2y_3^* - 3$$

Using $y_2^* = 0$ and $y_3^* = 0$ from line (ii) gives a consistent system with the unique solution $y_1^* = 3$. Therefore we must have $y^* = (3, 0, 0)$, and the corresponding slack vector is $z^* = (0, 1, 0)$.

(iv) Since $y^* \geq 0$ and $z^* \geq 0$, the vector y^* is indeed a dual minimizer corresponding to the given primal maximizer. That's no surprise: what we get from the work above is proof that *no other choice of* y^* *has this property.* That is, the minimizer in (D) is unique.

(e) Here we have a primal problem with a nonunique maximizer and a dual problem with a unique minimizer. Dual thinking reveals that *the given statement is false.* Indeed, we could rewrite (D) in standard primal form: it would still have a unique solution. But the dual of (D) would then be (P), whose solution is not unique.

Once more, with feeling: the Duality Theorem promises that if a LP has an optimizer, then its dual has an optimizer as well. But it does **not** assert that uniqueness of the optimizer on one side corresponds to uniqueness on the other, and this example illustrates that this is too much to hope for in general.

4. (a) We can standardize the primal problem by splitting each free variable into two nonnegative variables,

$$x_j = u_j - v_j, \quad u_j \geq 0,\ v_j \geq 0, \qquad j = 1, 2,$$

and replacing the equation constraint with a pair of opposing inequalities. The reformulated problem is

$$\begin{array}{lrcr}
\text{Maximize } F = & 6u_1 - 6v_1 + 6u_2 - 6v_2 - x_3 - x_4 & & \\
\text{subject to} & u_1 - v_1 + 2u_2 - 2v_2 + x_3 + x_4 & \leq & 5 \\
& 3u_1 - 3v_1 + u_2 - v_2 - x_3 & \leq & 8 \\
& u_2 - v_2 + x_3 + x_4 & \leq & 1 \\
& -u_2 + v_2 - x_3 - x_4 & \leq & -1 \\
& u_1, v_1, u_2, v_2, x_3, x_4 & \geq & 0
\end{array}$$

Let us use name the dual variables suggestively as y_1, y_2, r_3, and s_3. Then the dual becomes

$$\begin{array}{lrcr}
\text{Minimize } G = & 5y_1 + 8y_2 + r_3 - s_3 & & \\
\text{subject to} & y_1 + 3y_2 & \geq & 6 \\
& -y_1 - 3y_2 & \geq & -6 \\
& 2y_1 + y_2 + r_3 - s_3 & \geq & 1 \\
& -2y_1 - y_2 - r_3 + s_3 & \geq & -1 \\
& y_1 - y_2 + r_3 - s_3 & \geq & -1 \\
& y_1 + r_3 - s_3 & \geq & -1 \\
& y_1, y_2, r_3, s_3 \geq 0 & &
\end{array}$$

Here we notice two pairs of opposing inequalities, just like the ones that came up when we split the equation-constraint in the primal. Also, the dual variables r_3 and s_3 appear above only in the combination $r_3 - s_3$. By giving this combination the new name y_3, we arrive at a more compact form of the dual problem:

$$(D)\qquad \begin{array}{ll} \text{Minimize } g = 5y_1 + 8y_2 + y_3 & \\ \text{subject to} & y_1 + 3y_2 = 6 \\ & 2y_1 + y_2 + y_3 = 1 \\ & y_1 - y_2 + y_3 \geq -1 \\ & y_1 + y_3 \geq -1 \\ & y_1, y_2 \geq 0;\ y_3 \text{ free} \end{array}$$

(There is a general principle at work here. Equations in the primal problem will correspond to free variables in the dual, and free primal variables generate equations in the dual.)

(b) To test $x^* = (3, -1, 0, 2)$ for optimality, one option is to use the transformations in (a) to produce the extended vector

$$X^* = (u_1^*, v_1^*, u_2^*, v_2^*, x_3^*, x_4^*) = (3, 0, 0, 1, 0, 2)$$

and then to apply the complementarity recipe with the extended dual problem above. This turns out to be equivalent to a natural and intuitive modification of the recipe compatible with the general principles stated in (a). The speedy version works as follows.

(i) Primal Feasibility: The simple inequalities $x_3^* \geq 0$ and $x_4^* \geq 0$ both hold. The primal slack variables are

$$\begin{aligned} w_1^* &= 5 - x_1^* - 2x_2^* - x_3^* - x_4^* = 2 \\ w_2^* &= 8 - 3x_1^* - x_2^* + x_3^* = 0 \\ w_3^* &= 1 - x_2^* - x_3^* - x_4^* = 0 \end{aligned}$$

Since $w_1^* \geq 0$, $w_2^* \geq 0$ and $w_3^* = 0$, the vector x^* is indeed feasible in (P).

(ii) Complementarity I: Slack primal constraints imply zero dual variables. Since $w_1^* > 0$, we must have $y_1^* = 0$ in any minimizer y^* of (D).

(iii) Complementarity II: Nonzero primal decision values imply tight dual constraints. Since x_1^*, x_2^*, x_4^* are nonzero, constraints 1, 2, 4 in (D) must hold as equations. Remembering $y_1^* = 0$ from (ii), we obtain the 3×2 system

$$\begin{aligned} 3y_2^* &= 6 \\ y_2^* + y_3^* &= 1 \\ y_3^* &= -1 \end{aligned}$$

This looks overdetermined, but nonetheless has the unique solution $y_2^* = 2$, $y_3^* = -1$.

(iv) Dual feasibility: Steps (iii)–(iv) together suggest the dual vector $y^* = (0, 2, -1)$. This satisfies constraints numbered 1, 2, 4 in (D) by construction. However, constraint 3 fails: the dual slack

$$z_3^* = y_1^* - y_2^* + y_3^* + 1 = 0 - 2 - 1 + 1 = -2,$$

is negative! Therefore y^* is **not dual-feasible.**

Since the system of equations describing the vector y^* has no other solutions, the given vector x^* is **not optimal** in the primal.

Direct calculation (optional—not required for full credit) shows that each vector in the sequence

$$x^{(k)} = \left(\frac{7}{3} + 2k,\ 1 - 3k,\ 3k,\ 0\right), \qquad k = 1, 2, 3, \ldots,$$

is feasible in the given problem, and that $f(x^{(k)}) \to +\infty$ as $k \to \infty$. Thus the problem is unbounded. It follows that no vector can give a maximum: this rules out the particular x^* in the setup, along with all other candidates.

5. When (P) has a maximizer x^*, the duality theorem guarantees that (D) has a minimizer y^*. Together, x^* and y^* must satisfy the Complementary Slackness conditions. These involve the primal slack variables

$$w_1^* = 7 - x_1^* + 2x_2^* + x_3^* - x_4^*$$
$$w_2^* = 4 - x_1^* - 5x_2^* - x_3^* + x_4^*$$

Given $x_4^* = 1$, we have

$$w_1^* = 6 - x_1^* + 2x_2^* + x_3^*$$
$$w_2^* = 5 - x_1^* - 5x_2^* - x_3^*$$

Since x^* is feasible in (P), we have $w_2^* \geq 0$. In particular, since $x_2^* \geq 0$ and $x_3^* \geq 0$, we have $x_1^* \leq 5$. It follows that

$$w_1^* = 6 - x_1^* + 2x_2^* + x_3^* \geq 6 - x_1^* + 0 + 0 \geq 6 - 5 = 1.$$

Thus $w_1^* \neq 0$, so complementary slackness implies $y_1^* = 0$.

6. (a) The dual of the given problem is

$$(D) \quad \left[\begin{array}{ll} \text{minimize } g = & 5y_1 + 4y_2 \\ \text{subject to} & 3y_1 + 4y_2 \geq 8 \\ & 2y_1 - \ \ y_2 \geq 3 \\ & -2y_1 + \ \ y_2 \geq 4 \\ & y_1, y_2, y_3 \geq 0 \end{array}\right.$$

(b) If a vector (y_1, y_2, y_3) is feasible in (D), then it makes true inequalities out of the last two constraints, which add to give $0 \geq 7$. This is impossible, so problem (D) must be infeasible. Our code for this is $\min(D) = +\infty$.

(c) Problem (P) has a feasible origin, so $\max(P) \geq 0$. Certainly that means $\max(P) \neq -\infty$.

(d) In class, we showed that a discrepancy between $\min(D)$ and $\max(P)$ can only happen when $\min(D) = +\infty$ and $\max(P) = -\infty$. This scenario is incompatible with our observation in (c), so we must have $\max(P) = \min(D) = +\infty$. That is, problem (P) must be unbounded.

7. (a) If the farmer plants x_1 acres with corn, x_2 acres with soy, and x_3 acres with oats, her profit will be

$$\zeta = 40x_1 + 30x_2 + 25x_3.$$

Constraints on land, labour, and capital (respectively) lead to the inequalities in the following standard-form problem:

$$\begin{array}{lrcl} \text{maximize} & \zeta = 40x_1 + 30x_2 + 25x_3 & & \\ \text{subject to} & x_1 + x_2 + x_3 & \leq & 120 \\ & 6x_1 + 6x_2 + 3x_3 & \leq & 480 \\ & 36x_1 + 24x_2 + 18x_3 & \leq & 3600 \\ & x_1, x_2, x_3 \geq 0 & & \end{array}$$

(b) The farmer's plans to use all available land and labour imply $w_1^* = 0$ and $w_2^* = 0$ in the primal slack definitions

$$\begin{aligned} w_1 &= 120 - x_1 - x_2 - x_3 \\ w_2 &= 480 - 6x_1 - 6x_2 - 3x_3 \\ w_3 &= 3600 - 36x_1 - 24x_2 - 18x_3 \end{aligned}$$

The farmer's conjectured crop choices suggest a primal optimum $x^* \in \mathbb{R}^3$ in which $x_1^* > 0$, $x_2^* = 0$, and $x_3^* > 0$. Putting $x_2^* = 0$, $w_1^* = 0$, and $w_2^* = 0$ into the system above leads to

$$\begin{aligned} 0 = w_1^* &= 120 - x_1^* - x_3^* \\ 0 = w_2^* &= 480 - 6x_1^* - 3x_3^* \end{aligned}$$

This has the unique solution $x_1^* = 40$, $x_3^* = 80$, leading to $w_3^* = 720$.

We have a recipe for checking optimality of a proposed solution, and it requires familiarity with the dual problem.

$$\begin{array}{lrcl} \text{minimize} & \xi = 120y_1 + 480y_2 + 3600y_3 & & \\ \text{subject to} & y_1 + 6y_2 + 36y_3 & \geq & 40 \\ & y_1 + 6y_2 + 24y_3 & \geq & 30 \\ & y_1 + 3y_2 + 18y_3 & \geq & 25 \\ & y_1, y_2, y_3 \geq 0 & & \end{array}$$

(i) [Primal Feasibility] Here $x^* = (40, 0, 80) \geq (0, 0, 0)$ and $w^* = (0, 0, 720) \geq (0, 0, 0)$, so x^* is feasible in (P).

(ii) [Complementary Slackness, I] We already know $w_1^* = 0$ and $w_2^* = 0$, so the only new information comes from subscript 3: $w_3^* > 0$ requires $y_3^* = 0$.

(iii) [Complementary Slackness, II] We already know $x_2^* = 0$, so the new information comes from subscripts 1 and 3. Since $x_1^* > 0$ and $x_3^* > 0$, we must have $z_1^* = 0$ and $z_3^* = 0$, i.e.,

$$\begin{aligned} 0 = z_1^* &= y_1^* + 6y_2^* + 36y_3^* - 40 \\ 0 = z_3^* &= y_1^* + 3y_2^* + 18y_3^* - 25 \end{aligned}$$

Using $y_3^* = 0$ from part (ii) reduces this system to

$$\begin{aligned} y_1^* + 6y_2^* &= 40 \\ y_1^* + 3y_2^* &= 25 \end{aligned}$$

The unique solution is $y_1^* = 10$, $y_2^* = 5$. Therefore the only possible candidate for dual optimality is $y^* = (10, 5, 0)$. The corresponding dual slacks are $z^* = (0, z_2^*, 0)$, where

$$z_2^* = y_1^* + 6y_2^* + 24y_3^* - 30 = 10.$$

(iv) Since $y^* = (10, 5, 0) \geq (0, 0, 0)$ and $z^* = (0, 10, 0) \geq (0, 0, 0)$, vector y^* is dual-feasible.

Taken together, items (i)–(iv) confirm the primal optimality of $x^* = (40, 0, 80)$.

(c) Take $b^* = (b_1^*, b_2^*, b_3^*) \stackrel{\text{def}}{=} (120, 480, 3600)$ and let $Z(b)$ denote the farmer's maximum profit as a function of b. For inputs near b^*, we have

$$\left.\frac{\partial Z}{\partial b_1}\right|_{b^*} = y_1^* = 10, \qquad \left.\frac{\partial Z}{\partial b_2}\right|_{b^*} = y_2^* = 5, \qquad \left.\frac{\partial Z}{\partial b_3}\right|_{b^*} = y_3^* = 0.$$

These represent the rate at which small extra inputs boost the farmer's bottom line. The farmer will gain an advantage if she can buy more land for any price below \$10/acre or more labour for any price less than \$5/hour. More capital is worthless to her (\$0/dollar), because she already has leftover capital under her current plan ($w_3^* = 720$ dollars, to be precise). There is no profit to be gained by getting more.

8. (a) Since $x_1^* + x_2^* \leq 5$, we must have $x_1^* \leq 5$ and $x_2^* \leq 5$. This forces $x_3^* = 9$, which reduces the second constraint to

$$-x_1^* + 2x_2^* \leq -5.$$

One of the variables on the left must be 0, and it can't be x_1^*. Therefore $x_1^* = 5$ and $x_2^* = 0$. The only choice that works is $\mathbf{x}^* = (5, 0, 9)$.

(b) Here is the dual LP:

$$\begin{array}{ll} \min & 5y_1 + 4y_2 + 2y_3 \\ \text{subject to} & y_1 - \ \, y_2 + \ \, y_3 \geq 7 \\ & y_1 + 2y_2 \qquad\quad \geq 5 \\ & \qquad\ \ y_2 - \ \, y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

(c) Perhaps, by sheer luck or intelligent design, the vector in (a) is optimal. Let's check. (Drop the stars.)

(i) [Primal feasibility.] Clearly $(5, 0, 9) \geq (0, 0, 0)$. In addition,

$$\begin{aligned} w_1 &= 5 - x_1 - \ \ x_2 \qquad\quad = 0 \\ w_2 &= 4 + x_1 - 2x_2 - x_3 = 0 \\ w_3 &= 2 - x_1 \qquad\quad + x_3 = 6 \end{aligned}$$

These are all nonnegative, so $x = (5, 0, 9)$ is feasible.

(ii) [Complementary Slackness, I] Since $w_3 > 0$, we have $y_3 = 0$.

(iii) [Complementary Slackness, II] Since $x_1 > 0$ and $x_3 > 0$, we must have $z_1 = 0$ and $z_3 = 0$, where z_j is the surplus variable for dual constraint number j. Taken together with $y_3 = 0$, we have the system

$$\begin{aligned} 0 = z_1 &= y_1 - y_2 - 7 \\ 0 = z_3 &= \qquad\ \ y_2 - 2 \end{aligned}$$

The second equation gives $y_2 = 2$; the first then says $y_1 = 9$.

(iv) [Dual Feasibility] The dual vector $y = (9, 2, 0)$ obeys $z_1 = 0$ and $z_3 = 0$ by construction in part (iii), and

$$z_2 = y_1 + 2y_2 - 5 = 8 \geq 0.$$

Therefore y is dual-feasible.

The steps above confirm that $x = (5, 0, 9)$ is optimal for the primal problem and that $y = (9, 2, 0)$ is optimal for the dual. For any primal maximizer $\mathbf{x}^*$, *every dual solution* must be generated by the procedure we have shown. Uniqueness at every step guarantees that $y = (9, 2, 0)$ is the unique solution for the dual problem.

(d) The procedure just described works backwards, too. Starting with the dual solution $y = (9, 2, 0)$, the strict surplus $z_2 = 8 > 0$ requires $x_2 = 0$. Then, from $y_1 > 0$ and $y_2 > 0$, we get the pair of primal requirements

$$0 = w_1 = 5 - x_1$$
$$0 = w_2 = 4 + x_1 - x_3$$

These offer no alternatives: we are compelled to choose $x_1 = 5$ and deduce $x_3 = 9$. Thus the primal maximizer is also unique.

Note: It is not obvious that uniqueness of the dual minimizer guarantees uniqueness of the primal maximizer. Question 1 provides a counterexample. So part (d) of this exercise is not vacuous ... some nontrivial argument is required. (A well-formulated appeal to the nondegeneracy of the dual minimizer provides an attractive alternative to the approach above.)