Math 340(921) Problem Set 4

Due in class on Friday 7 June 2013

1. (a) Let S be the set in $\mathbb{R}^2$ defined by these simultaneous linear inequalities:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 - \sqrt{3}\,x_2 \geq -2\sqrt{3}, \quad x_1 + \sqrt{3}\,x_2 \geq \sqrt{3}, \quad x_1 + x_2 \leq 3\sqrt{3}.$$

Make a careful sketch of S.

(b) Find the coordinates of all the corner points of S.

(c) Find the exact interior angle (in radians) at each corner point of S. Any logically correct method is acceptable.

(d) Find the point of S that gives the largest possible value to $\zeta(x_1, x_2) = x_1 - x_2$. Label this point A on your sketch from part (a); include its exact coordinates on the figure. Working counterclockwise around S, label the other corner points $B, C, \ldots$; give their coordinates, too.

(e) For each nonzero vector $\mathbf{c} = (c_1, c_2)$, find all points in S that maximize the function $\zeta(x) = c^T x$. Present your results both algebraically and graphically, using a "map" drawn on the (c_1, c_2)-plane.

(f) Let $Z = Z(c)$ denote the maximum value of $c^T x$ when $x \in S$, expressed as a function of the vector c. In each region of the (c_1, c_2)-plane described in part (e), there is a simple formula for $Z(c)$. Find all such formulas and the regions where they apply.

2. Consider the following LP:

$$\begin{aligned}
\text{maximize} \quad & \zeta = -x_1 - 2x_2 \\
\text{subject to} \quad & -2x_1 + 7x_2 \leq 6 \\
& -3x_1 + x_2 \leq -1 \\
& 9x_1 - 4x_2 \leq 6 \\
& x_1 - x_2 \leq 1 \\
& 7x_1 - 3x_2 \leq 6 \\
& -5x_1 + 2x_2 \leq -3 \\
& x_1, x_2 \geq 0
\end{aligned}$$

(a) Write down the dual problem.

(b) Transform the dual to standard form and solve it using the simplex method. Show the dictionaries for each pivot step.

(c) Find the solution for the given (primal) problem.

Credit: Vanderbei problem 5.6.

3. The following dictionary has been produced by solving a certain linear program in standard form:

$$\begin{array}{rl}
f = & 15 - x_2 - 3x_4 \\
\hline
x_1 = & 1 + x_2 - 2x_4 + x_6 \\
x_3 = & 3 - 4x_2 + 3x_4 - 2x_6 \\
x_5 = & 2 + 3x_2 + 2x_4
\end{array}$$

More . . .

(a) Find the original problem—call it (P).

(b) Find all maximizing points in problem (P).

(c) The client that brought you this problem wants to see a maximum value of 17. Suppose you can change the number on the right side of exactly one of the constraints in the original problem. Which constraint will you choose to modify, and by how much, in your first attempt to satisfy the client? Explain. (A well-informed first approximation will suffice.)

(d) Write (D), the dual problem corresponding to (P), and find all minimizing inputs for (D).

(e) Decide if the following statement about uniqueness is true or false. Explain your reasons.

> "*In linear programming, if the primal problem has a unique maximizer then the dual problem must have a unique minimizer.*"

4. Consider the following problem. Notice that it is not in standard symmetric form.

$$(P)\qquad \begin{aligned} \text{Maximize } & 6x_1 + x_2 - x_3 - x_4 \\ \text{subject to } & x_1 + 2x_2 + x_3 + x_4 \le 5 \\ & 3x_1 + x_2 - x_3 \le 8 \\ & x_2 + x_3 + x_4 = 1 \\ & x_3, x_4 \ge 0;\ x_1, x_2 \text{ free} \end{aligned}$$

(a) Write the dual problem.

(b) Decide if $x^* = (3, -1, 0, 2)$ is a true maximizer in (P). If so, explain why and find a solution for the dual problem; if not, explain why not. [Credit: Chvátal text, problem 9.5.]

5. In a certain standard-form linear programming problem (P), the first two constraints are

$$\begin{aligned} x_1 - 2x_2 - x_3 + x_4 &\le 7 \\ x_1 + 5x_2 + x_3 - x_4 &\le 4 \end{aligned}$$

A maximizer x^* is known to exist, and to have $x_4^* = 1$. Prove that there must exist a dual minimizer y^* satisfying $y_1^* = 0$.

6. Prove that the following standard-form LP is unbounded by following the steps below.

$$(P)\qquad \left[\begin{aligned} \text{maximize } & f = 8x_1 + 3x_2 + 4x_3 \\ \text{subject to } & 3x_1 + 2x_2 - 2x_3 \le 5 \\ & 4x_1 - x_2 + x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}\right.$$

(a) Write the dual problem, (D).

(b) Study the constraints in (D). Explain why $\min(D) = +\infty$. [Try to avoid the simplex method.]

(c) Explain why $\max(P) \ne -\infty$.

(d) Explain how the desired result follows from your findings in parts (b) and (c).

7. A farmer owns 120 acres of land which are capable of producing corn, soybeans and oats. The following table represents the requirements for labour and capital per acre and the net profit per acre for each crop:

Crop	Labour (hrs)	Capital ($)	Net profit ($)
corn	6	36	40
soybeans	6	24	30
oats	3	18	25

The farmer has available $3600 of capital and 480 hours of labour for these crops.

(a) Formulate a linear programming problem for determining how the farmer should maximize her profit.

(b) The farmer suspects that she will maximize profits by using all available land and labour, planting corn and oats but no soybeans. Verify this using complementary slackness. How much should be planted with each crop?

(c) How much extra should the farmer be willing to pay for small additional amounts of land, labour and capital?

Credit: Professor Israel's MATH 340 exam from April 2004.

8. Consider this LP:

$$\begin{array}{ll} \max & 7x_1 + 5x_2 + 2x_3 \\ \text{subject to} & x_1 + x_2 \leq 5 \\ & -x_1 + 2x_2 + x_3 \leq 4 \\ & x_1 - x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(a) Use the entries 0, 5, and 9, in some order, to invent a feasible input $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*)$.

(b) Find the dual linear program.

(c) Find all solutions of the dual LP. Explain how you know there are no others.
(*Clue*: Part (a) may help.)

(d) Find all solutions of the given (primal) LP. Explain how you know there are no others.