M340(921) Solutions—Problem Set 3

1. Phase One: First rewrite the second inequality constraint in standard form, then build the auxiliary problem

$$\begin{array}{llr} \text{maximize} & w = & -x_0 \\ \text{subject to} & & 2x_1 + x_2 + x_3 - x_0 \le 2, \\ & & -3x_1 - 4x_2 - 2x_3 - x_0 \le -8, \\ & & x_1, x_2, x_3, x_0 \ge 0. \end{array}$$

A dictionary for this problem is

$$D_0: \quad \begin{array}{rl} x_4 = & 2 - 2x_1 - x_2 - x_3 + x_0 \\ x_5 = & -8 + 3x_1 + 4x_2 + 2x_3 + x_0 \\ \hline w = & -x_0. \end{array}$$

The worst infeasibility here can be addressed by pivoting x_0 into the basis and x_5 out. The pivot equation is $x_0 = 8 - 3x_1 - 4x_2 - 2x_3 + x_5$. This leads to the feasible dictionary

$$D_1: \quad \begin{array}{rl} x_4 = & 10 - 5x_1 - 5x_2 - 3x_3 + x_5 \\ x_0 = & 8 - 3x_1 - 4x_2 - 2x_3 + x_5 \\ \hline w = & -8 + 3x_1 + 4x_2 + 2x_3 - x_5. \end{array}$$

Now x_2 enters and x_0 leaves (there is a tie between x_0 and x_4: we select x_0 because we want it to be nonbasic). The pivot equation is $x_2 = (8 - 3x_1 - x_0 - 2x_3 + x_5)/4$. It leads to

$$D_2: \quad \begin{array}{rl} x_4 = & 0 - (5/4)x_1 + (5/4)x_0 - (1/2)x_3 - (1/4)x_5 \\ x_2 = & 2 - (3/4)x_1 - (1/4)x_0 - (1/2)x_3 + (1/4)x_5 \\ \hline w = & - x_0. \end{array}$$

This shows that the maximum value in the auxiliary problem equals 0, which means that there does exist a basic feasible solution for the original constraint system. The dictionary for this situation is formed by simply dropping all mention of x_0 from the previous dictionary, and re-calibrating the original objective function to match:

$$f = 3x_1 + 2x_2 + 3x_3 = 3x_1 + 2(2 - \frac{3}{4}x_1 - \frac{1}{2}x_3 + \frac{1}{4}x_5) + 3x_3 = 4 + \frac{3}{2}x_1 + 2x_3 + \frac{1}{2}x_5.$$

Phase Two: Work on the original problem, starting with the feasible dictionary

$$D_3: \quad \begin{array}{rl} x_4 = & 0 - (5/4)x_1 - (1/2)x_3 - (1/4)x_5 \\ x_2 = & 2 - (3/4)x_1 - (1/2)x_3 + (1/4)x_5 \\ \hline f = & 4 + (3/2)x_1 + 2x_3 + (1/2)x_5. \end{array}$$

Let x_3 enter the basis and x_4 leave (a degenerate pivot):

$$D_4: \quad \begin{array}{rl} x_3 = & 0 - (5/2)x_1 - 2x_4 - (1/2)x_5 \\ x_2 = & 2 + (1/2)x_1 + x_4 + (1/2)x_5 \\ \hline f = & 4 - (7/2)x_1 - 4x_4 - (1/2)x_5. \end{array}$$

This is an optimal dictionary. It shows that the original problem has $f_{\text{MAX}} = 4$, attained at $(x_1, x_2, x_3) = (0, 2, 0)$.

2.

(a) Every feasible solution (x_1, x_2, x_3) has $x_1 \le 2$, so $2x_1 \le 4$. Together with the first constraint, this implies

$$f = 2x_1 + (3x_1 + x_2 - x_3) \le 4 + (-2) = 2.$$

(Another approach is to write the dual problem and show that it has a feasible solution. This shows $\min(D) < +\infty$; since $\max(P) \le \min(D)$ in general, it follows that $\max(P) < +\infty$, as required.)

(b) Just staring at the constraints suggests the point $(x_1, x_2, x_3) = (0, 0, 2)$. This observation gets full marks, but leaves more work for part (c).

Recall that a "basic solution" is one that can be expressed using a dictionary. The given problem has $m = 3$ constraints, so there will be 3 constraint rows in the dictionary, so a BFS can have at most 3 nonzero values in the list of decision and slack variables. Thus the decision vector $\mathbf{x} = (1, 0, 5)$ is feasible but not basic, because the slack vector $\mathbf{s} = (0, 4, 1)$ has too many nonzero entries. By contrast, $\mathbf{x} = (2, 0, 4.5)$ is a BFS because $\mathbf{s} = (-1.5, 0, 0)$ has only one nonzero element.

A more systematic approach is to introduce an auxiliary variable $x_0 \ge 0$ and work on the "Phase One" problem

$$\begin{array}{lrcl}
\text{maximize} & g = & & -x_0 \\
\text{subject to} & 3x_1 + x_2 - x_3 - x_0 & \le & -2 \\
 & 3x_1 - x_2 - 2x_3 - x_0 & \le & -3 \\
 & x_1 \qquad\qquad - x_0 & \le & 2 \\
 & x_0, x_1, x_2, x_3 \ge 0 & &
\end{array}$$

This gives an infeasible initial dictionary:

$$\begin{array}{rcl}
x_4 & = & -2 - 3x_1 - x_2 + x_3 + x_0 \\
x_5 & = & -3 - 3x_1 + x_2 + 2x_3 + x_0 \\
x_6 & = & 2 - x_1 + x_0 \\
\hline
f & = & -x_0
\end{array}$$

Pivot in x_0 and pivot out x_5. This gives the first feasible dictionary:

$$\begin{array}{rcl}
x_4 & = & 1 - 2x_2 - x_3 + x_5 \\
x_0 & = & 3 + 3x_1 - x_2 - 2x_3 + x_5 \\
x_6 & = & 5 + 2x_1 - x_2 - 2x_3 + x_5 \\
\hline
f & = & -3 - 3x_1 + x_2 + 2x_3 - x_5
\end{array}$$

Now the largest-coefficient rule selects x_3 to enter and x_4 to leave:

$$\begin{array}{rcl}
x_3 & = & 1 - 2x_2 - x_4 + x_5 \\
x_6 & = & 3 + 2x_1 + 3x_2 + 2x_4 - x_5 \\
x_0 & = & 1 + 3x_1 + 3x_2 + 2x_4 - x_5 \\
\hline
f & = & -1 - 3x_1 - 3x_2 - 2x_4 + x_5
\end{array}$$

Now x_5 must enter, and x_0 must leave. This pivot solves the auxiliary problem:

$$\begin{array}{l} x_3 = 2 + 3x_1 + \ \ x_2 + \ \ x_4 - x_0 \\ x_6 = 2 - \ \ x_1 \qquad\qquad\quad + x_0 \\ x_5 = 1 + 3x_1 + 3x_2 + 2x_4 - x_0 \\ \hline f \ = \qquad\qquad\qquad\qquad - x_0 \end{array}$$

A Basic Feasible Solution for the original problem is $(0, 0, 2)$.

(c) In terms of the selected basic variables, the original objective is

$$f = 5x_1 + x_2 - x_3 = 5x_1 + x_2 - (2 + 3x_1 + x_2 + x_4) = -2 + 2x_2 - x_4.$$

Therefore a feasible dictionary for the original problem is

$$\begin{array}{l} x_3 = \quad 2 + 3x_1 + \ \ x_2 + \ \ x_4 \\ x_6 = \quad 2 - \ \ x_1 \\ x_5 = \quad 1 + 3x_1 + 3x_2 + 2x_4 \\ \hline f \ = -2 + 2x_1 \qquad\quad - \ \ x_4 \end{array}$$

Pivot x_1 into the basis and x_6 out to attain optimality:

$$\begin{array}{l} x_3 = 8 - 3x_1 + \ \ x_2 + \ \ x_4 \\ x_1 = 2 - \ \ x_6 \\ x_5 = 7 - 3x_6 + 3x_2 + 2x_4 \\ \hline f \ = 2 - 2x_6 \qquad\quad - \ \ x_4 \end{array}$$

The maximum value is 2, and a Basic Optimal Solution is $(2, 0, 8)$.

The maximizer is not unique. Any value of $x_2 \geq 0$ compatible with the constraints is allowed. But none of the basic variables become negative as x_2 increases, so there is no upper limit to the possible size of x_2. The complete set of solutions is the family

$$(x_1, x_2, x_3) = (2, x_2, 8 + x_2), \qquad x_2 \geq 0.$$

(Notice that the set of maximizers is unbounded, but the objective value is not. So this is not an "unbounded problem".)

3. The handout entitled "One Primal Simplex Pivot" provides convenient notation for this task.

(a) Recall the expanded form of dictionary D^0 in which the entering index E is highlighted:

$$\begin{array}{l} f = v^0 + \ c_E^0 x_E + \displaystyle\sum_{j \in \mathcal{N}^0 \setminus \{E\}} c_j^0 x_j \\ \hline x_i = b_i^0 - a_{iE}^0 x_E - \displaystyle\sum_{j \in \mathcal{N}^0 \setminus \{E\}} a_{ij}^0 x_j \qquad (i \in \mathbb{B}^0) \end{array} \tag{3}$$

We have $b_E^0 \geq 0$ because the dictionary is feasible, and $c_E^0 > 0$ because index E is eligible to enter the basis. Also, since L is eligible to leave, $a_{LE}^0 > 0$.

The pivot equation in D^0 comes from line $i = L$:

$$x_L = b^0_L - a^0_{LE} x_E - \sum_{j \in \mathcal{N}^0 \setminus \{E\}} a^0_{Lj} x_j.$$

Rearranging it (knowing that $a^0_{LE} > 0$ helps us here) gives

$$x_E = \frac{1}{a^0_{LE}} \left(b^0_L - x_L - \sum_{j \in \mathcal{N}^0 \setminus \{E\}} a^0_{Lj} x_j \right), \quad \text{so} \quad c^0_E x_E = \frac{c^0_E}{a^0_{LE}} \left(b^0_L - x_L - \sum_{j \in \mathcal{N}^0 \setminus \{E\}} a^0_{Lj} x_j \right).$$

Plugging this back into the objective row of D^0 gives the objective row of D^+:

$$f = v^0 + \frac{c^0_E b^0_L}{a^0_{LE}} - \left(\frac{c^0_E}{a^0_{LE}} \right) x_L + \sum_{j \in \mathcal{N}^0 \setminus \{E\}} \left(c^0_j - a^0_{Lj} \left(\frac{c^0_E}{a^0_{LE}} \right) \right) x_j.$$

Check: the new nonbasic indices are $\mathcal{N}^+ = \left(\mathcal{N}^0 \setminus \{E\}\right) \cup \{L\}$, and each gets mentioned exactly once on the right side here.

(b) In the updated dictionary D^+, the coefficient of x_L is $c^+_L = -c^0_E / a^0_{LE}$. We know both $c^0_E > 0$ and $a^0_{LE} > 0$ (see part (a)), so we have $c^+_L < 0$. Therefore index L is not an eligible entering index in dictionary D^+.

4. Suppose we decide to blend x_1 tons of Carol's Road Mix, x_2 tons of Gord's Grits, and x_3 tons of pure salt into each ton of highway mixture. (Clearly each $x_i \geq 0$.) The constraints become

$$\begin{array}{ll}
\text{total mass involved:} & x_1 + x_2 + x_3 = 1.00 \\
\text{mass of sand (tons):} & 0.75x_1 + 0.60x_2 \leq 0.70 \\
\text{mass of salt (tons):} & 0.02x_1 + 0.06x_2 + 1.00x_3 \geq 0.10 \\
\text{cost of ingredients:} & f = 50x_1 + 120x_2 + 800x_3
\end{array}$$

Using the first equation, we can eliminate x_3 in favour of this pair of conditions:

$$x_3 = 1 - x_1 - x_2, \qquad x_1 + x_2 \leq 1. \tag{$*$}$$

The cost of ingredients becomes $f = 800 - 750x_1 - 680x_2$; we can minimize this by maximizing $\zeta = 800 - f$. Substituting $(*)$ into the constraints above leads to a standard problem in just two variables:

$$\begin{array}{ll}
\text{maximize} & \zeta = 750x_1 + 680x_2 \\
\text{subject to} & 0.75x_1 + 0.60x_2 \leq 0.70 \\
& 0.98x_1 + 0.94x_2 \leq 0.90 \\
& x_1 + x_2 \leq 1.00 \\
& x_1, x_2 \geq 0
\end{array}$$

With the computer on our team, only some typing is left to do.

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Optimal Solution:  z = 688.776; x1 = 0.918367, x2 = 0
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We must do our own calculation of $x_3 = 1 - 0.91837 - 0 = 0.08163$ to complete the report.

Summary. The cheapest mixture combines 91.84% of Carol's Road Mix with 8.16% of pure salt. Its cost per ton is

$$f_{\min} = 800 - \zeta_{\max} = \$111.22.$$

5. Suppose Sam buys x_1, x_2, and x_3 fans in April, May, and June, respectively, and sells w_1, w_2, and w_3 fans in these months. The supply and demand constraints translate into

$$x_i \le 450, \quad w_i \le 600, \qquad i = 1, 2, 3. \tag{$*$}$$

Sam's cost for fans and revenue from sales are easy:

$$\begin{array}{ll} \text{total fan cost:} & 31x_1 + 33x_2 + 36x_3, \\ \text{total revenue from sales:} & 40w_1 + 44w_2 + 48w_3. \end{array}$$

The number of fans unsold at the end of April is $h_1 \overset{\text{def}}{=} x_1 - w_1$; the number unsold at the end of May is $h_2 \overset{\text{def}}{=} (x_1 + x_2) - (w_1 + w_2)$. These must both be not larger than 300, i.e.,

$$x_1 - w_1 \le 300 \qquad \text{and} \qquad x_1 + x_2 - w_1 - w_2 \le 300.$$

They also help determine Sam's

$$\text{total storage costs:} \quad 2(x_1 - w_1) + 2(x_1 + x_2) - 2(w_1 + w_2) = 4x_1 + 2x_2 - 4w_1 - 2w_2.$$

The number of fans unsold at the end of June (h_3) must be 0. This gives

$$0 = (x_1 + x_2 + x_3) - (w_1 + w_2 + w_3), \quad \text{i.e.,} \quad w_3 = x_1 + x_2 + x_3 - w_1 - w_2. \tag{$**$}$$

Using ($**$), we can eliminate w_3 throughout all the developments above, provided we continue to enforce $0 \le w_3 \le 600$ by a pair of constraints.

Sam's business issues lead to this standard problem:

$$\begin{array}{lrrrrrl} \text{maximize} & \multicolumn{6}{l}{\zeta = 13x_1 + 13x_2 + 12x_3 - 4w_1 - 2w_2} \\ \text{subject to} & x_1 & & & - \; w_1 & & \le 300 \\ & x_1 + & x_2 & & - \; w_1 - & w_2 & \le 300 \\ & -x_1 - & x_2 - & x_3 + & w_1 + & w_2 & \le 0 \\ & x_1 + & x_2 + & x_3 - & w_1 - & w_2 & \le 600 \\ & x_1 & & & & & \le 450 \\ & & x_2 & & & & \le 450 \\ & & & x_3 & & & \le 450 \\ & & & & w_1 & & \le 600 \\ & & & & & w_2 & \le 600 \\ & \multicolumn{6}{l}{x_1, x_2, x_3, w_1, w_2 \ge 0} \end{array}$$

The online solver crunches through this rapidly, giving

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Optimal Solution:  z = 15300; x1 = 450, x2 = 450, x3 = 450, w1 = 150, w2 = 600
```

Line ($**$) gives $w_3 = 600$; storage amounts (from above) are $h_1 = 300$, $h_2 = 150$, $h_3 = 0$.

The following strategy realizes Sam's maximum profit of \$15300:

	Buy	**Sell**	**Hold**
Apr	450	150	300
May	450	600	150
Jun	450	600	0

6. (a) Define $y_2 = x_2 + 6$ and $y_3 = x_3 - 1$. With these choices, we have

$$x_2 \geq -6 \iff y_2 \geq 0 \qquad \text{and} \qquad x_3 \geq 1 \iff y_3 \geq 0.$$

Substituting $x_2 = y_2 - 6$ and $x_3 = y_3 + 1$ puts the problem into the form

$$\begin{array}{ll} \text{minimize} & f = 129 - 6x_1 + 2y_2 - 9y_3 \\ \text{subject to} & 2x_1 - 6y_2 - y_3 \leq -25 \\ & x_1 + y_2 + 9y_3 \leq 17 \\ & x_1 \qquad\qquad\quad \leq 5 \\ & x_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0 \end{array}$$

The points minimizing f are the same ones that maximize $\zeta = 129 - f$. The desired standard-form problem is

$$\begin{array}{ll} \text{maximize} & f = 6x_1 - 2y_2 + 9y_3 \\ \text{subject to} & 2x_1 - 6y_2 - y_3 \leq -25 \\ & x_1 + y_2 + 9y_3 \leq 17 \\ & x_1 \qquad\qquad\quad \leq 5 \\ & x_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0 \end{array}$$

(b) The online simplex tool mentioned on the course web page finds

$$\zeta_{\max} = 24.8491 \quad \text{at} \quad (x_1, y_2, y_3) = (5.0000, 5.7169, 0.69811).$$

(c) Reversing the substitutions in part (a) gives

$$f_{\min} = 129 - \zeta_{\max} = 104.15 \quad \text{at} \quad (x_1, x_2, x_3) = (5.0000, -0.2831, 1.6981).$$

7. (a) Let x_1, x_2, and x_3 denote the number of Friendship, Romance, and Forgiveness bouquets the florist makes. Each of these must be nonnegative. The gross receipts from selling all types of arrangements will be

$$r = 5.5x_1 + 10.5x_2 + 13x_3.$$

The supply of roses makes some calculations simple.

$$\begin{array}{ll} \text{Number of roses used:} & w_1 = 2x_1 + 6x_2 + 4x_3, \\ \text{75 dozen roses available:} & w_1 \leq 900 \\ \text{Cost of roses used (\$):} & \gamma_1 = (14.40/12)w_1 = 2.4x_1 + 7.2x_2 + 4.8x_3. \end{array}$$

The carnation situation is more delicate. Let x_4 denote the number of expensive imported carnations used. The supply constraints give

$$\begin{array}{ll} \text{Number of carnations used:} & w_2 = 5x_1 + 3x_2 + 12x_3, \\ \text{Available carnations (2 suppliers):} & w_2 \leq 1020 + x_4, \quad x_4 \leq 780. \end{array}$$

The cost of carnations is more complicated. The cheap local ones cost \$5.40/12 = \$0.45 each, and the expensive imported ones cost \$9.00/12 = \$0.75 each. That's a surcharge of \$0.30 for the imported ones. Therefore

$$\text{Cost of carnations used (\$):} \quad \gamma_2 = 0.45w_2 + 0.30x_4.$$

The profit in this business is the selling price minus the material costs, i.e., $\zeta = r - \gamma_1 - \gamma_2$. Eliminating w_1 and w_2 using the equations above, we arrive at this standard-form problem:

$$\begin{array}{ll} \text{maximize} & \zeta = 0.85x_1 + 1.95x_2 + 2.80x_3 - 0.30x_4 \\ \text{subject to} & 2x_1 + 6x_2 + 4x_3 \leq 900 \\ & 5x_1 + 3x_2 + 12x_3 - x_4 \leq 1020 \\ & x_4 \leq 780 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

(b) The online solver at `http://www.zweigmedia.com/RealWorld/simplex.html` responds well to the following input:

```
maximize z = 0.85 x1 + 1.95 x2 + 2.80 x3 - 0.30 x4
5 x1 + 3 x2 + 12 x3 - x4  <= 1020
2 x1 + 6 x2 +  4 x3       <=  900
                     x4  <=  780
```

Its one-line response is

```
Optimal Solution:  z = 378; x1 = 0, x2 = 112, x3 = 57, x4 = 0
```

(c) The optimal business plan is to make no Friendship bouquets, 112 Romance arrangements, and 57 Forgiveness bouquets. This requires no carnations at all from Surrey; from the local suppliers the florist should order

$$\begin{aligned} w_1 &= 2(0) + 6(112) + 4(57) = 900 \text{ roses (that's 75 dozen)}, \\ w_2 &= 5(0) + 3(112) + 12(57) = 1020 \text{ carnations (that's 85 dozen)}. \end{aligned}$$

Her profit will be \$378.00.

(d) If a dozen carnations from Surrey cost only \$1.20 more than a dozen carnations found locally, the surcharge for using each imported stem is only \$0.10. This changes the coefficient of x_4 in the objective function from -0.30 to -0.10. The new computed result is

```
Optimal Solution:  z = 417; x1 = 0, x2 = 60, x3 = 135, x4 = 780
```

Now the florist can achieve a higher profit, \$417.00, by making 0 Friendship bouquets, 60 Romance bouquets, and 135 Forgiveness arrangements. This will consume

$$\begin{aligned} w_1 &= 2(0) + 6(60) + 4(135) = 900 \text{ roses (that's 75 dozen)}, \\ w_2 &= 5(0) + 3(60) + 12(135) = 1800 \text{ carnations (that's } 85 + 65 \text{ dozen)}. \end{aligned}$$

All 65 dozen carnations from Surrey will be required.