M340(921) Solutions—Problem Set 2

1. Use the Simplex Method with dictionaries to solve textbook problem 2.1(a), page 26. Show all your work, then write a summary giving the optimal value, the optimal solution, and the sequence of feasible basic solutions that the Simplex Method visits on its way to the maximizing point.

Problem statement:

$$\begin{aligned} \text{Maximize } f &= 3x_1 + 2x_2 + 4x_3 \\ \text{subject to } \quad x_1 + \ x_2 + 2x_3 &\le 4 \\ 2x_1 \qquad + 3x_3 &\le 5 \\ 2x_1 + \ x_2 + 3x_3 &\le 7 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Introduce slack variables $w_1 = x_4$, $w_2 = x_5$, $w_3 = x_6$ to define this initial dictionary:

$$\begin{aligned} x_4 &= 4 - x_1 - x_2 - 2x_3 \\ x_5 &= 5 - 2x_1 \qquad - 3x_3 \\ x_6 &= 7 - 2x_1 - x_2 - 3x_3 \\ \hline f &= 0 + 3x_1 + 2x_2 + 4x_3 \end{aligned}$$

Any nonbasic variable could enter the basis and improve the payoff, but the largest coefficient inspires us to select x_3 to enter. The largest permitted value for x_3 is $5/3$: this will make $x_5 = 0$, so the leaving variable must be x_5. The pivot equation is

$$3x_3 = 5 - 2x_1 - x_5, \qquad \text{or} \qquad x_3 = (5/3) - (2/3)x_1 - (1/3)x_5.$$

Back-substitution leads to the following new dictionary:

$$\begin{aligned} x_3 &= (5/3) - (2/3)x_1 \qquad - (1/3)x_5 \\ x_4 &= (2/3) + (1/3)x_1 - x_2 + (2/3)x_5 \\ x_6 &= 2 \qquad\qquad - x_2 + x_5 \\ \hline f &= (20/3) + (1/3)x_1 + 2x_2 - (4/3)x_5 \end{aligned}$$

Now either of x_1 or x_2 could enter, but x_2 has the largest coefficient, so let's pick that one. Increasing x_2 has no influence on x_3 or x_6, but it will spoil the nonnegativity of x_4 unless we choose $x_2 \le 2/3$. This identifies the leaving variable: x_4. The pivot equation is

$$x_2 = (2/3) + (1/3)x_1 - x_4 + (2/3)x_5.$$

Back-substitution leads to the following new dictionary:

$$\begin{aligned} x_2 &= (2/3) + (1/3)x_1 - x_4 + (2/3)x_5 \\ x_3 &= (5/3) - (2/3)x_1 \qquad - (1/3)x_5 \\ x_6 &= (4/3) - (1/3)x_1 + x_4 + (1/3)x_5 \\ \hline f &= 8 + x_1 - 2x_4 \end{aligned}$$

Now increasing x_4 will decrease the payoff, increasing x_5 will make no difference, but increasing x_1 will be rewarding. So choose x_1 to enter the basis. The rows involving x_3 and x_6 provide restrictions on x_1, and the stricter of these is provided by x_3. So x_3 leaves, via

$$(2/3)x_1 = (5/3) - x_3 - (1/3)x_5, \qquad \text{or} \qquad x_1 = (5/2) - (3/2)x_3 - (1/2)x_5.$$

The resulting dictionary is

$$\begin{array}{rl} x_1 = & (5/2) - (3/2)x_3 \qquad\quad - (1/2)x_5 \\ x_2 = & (3/2) - (1/2)x_3 - \ \ x_4 + (1/2)x_5 \\ x_6 = & (1/2) + (1/2)x_3 + \ \ x_4 + (1/2)x_5 \\ \hline f \ = & (21/2) - (3/2)x_3 - 2x_4 - (1/2)x_5 \end{array}$$

Now any feasible assignment to the nonbasic variables other than 0 clearly makes our payoff worse, so we have found the problem's unique maximizer: $f_{\text{MAX}} = (21/2)$, attained at

$$(x_1^*, x_2^*, x_3^*) = \left(\frac{5}{2}, \frac{3}{2}, 0\right).$$

Summary. Here are the steps presented in detail above.

Setup: $\mathcal{B} = \{4, 5, 6\}$, $\mathbf{x} = (0, 0, 0, 4, 5, 7)$, $f(\mathbf{x}) = 0$.
After Pivot 1: $\mathcal{B} = \{3, 4, 6\}$, $\mathbf{x} = (0, 0, 5/3, 2/3, 0, 2)$, $f(\mathbf{x}) = 20/3$.
After Pivot 2: $\mathcal{B} = \{2, 3, 6\}$, $\mathbf{x} = (0, 2/3, 5/3, 0, 0, 4/3)$, $f(\mathbf{x}) = 8$.
After Pivot 3: $\mathcal{B} = \{1, 2, 6\}$, $\mathbf{x} = (5/2, 3/2, 0, 0, 0, 1/2)$, $f(\mathbf{x}) = 21/2$.

2. (a) Use the simplex method to solve the following problem:

$$\begin{array}{ll} \text{maximize} & f = 4x_1 + 2x_2 + 2x_3 \\ \text{subject to} & x_1 + 3x_2 - 2x_3 \leq 3, \\ & 4x_1 + 2x_2 \leq 4, \\ & x_1 + x_2 + x_3 \leq 2, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

Use Anstee's pivot-selection rules; report the maximum value and the point that attains it.

(b) Suppose the objective function f in the part (a) is replaced with

$$g = 3x_1 + 2x_2 + x_3.$$

Adjust the optimal final dictionary from part (a) to produce a feasible dictionary for the g-problem. Then answer these questions. Is the maximizing point from part (a) still a maximizer for g? If so, explain why; if not, find the new maximum value and all points that achieve it.

(c) [Much like (b).] Suppose the objective function f in part (a) is replaced with

$$h = 4x_1 + 4x_2.$$

Adjust the optimal final dictionary from part (a) to produce a feasible dictionary for the h-problem. Is the maximizing point from part (a) still a maximizer for h? If so, explain why; if not, find the new maximum value and all points that achieve it.

Discussion: Parts (b)–(c) give you personal experience of re-using work you have already done to save time when solving a new problem that is pretty similar to a known one.

(a) We have three constraints, so we define three slack variables. The definitions form the first three lines of the initial dictionary:

$$\begin{array}{l} x_4 = 3 - x_1 - 3x_2 + 2x_3 \\ x_5 = 4 - 4x_1 - 2x_2 \\ x_6 = 2 - x_1 - x_2 - x_3 \\ \hline f = 4x_1 + 2x_2 + 2x_3 \end{array}$$

The largest coefficient in the objective row is attached to x_1, so this variable will enter the basis. Keeping $x_2 = 0$ and $x_3 = 0$, the increase in x_1 is limited most strictly by the inequality $x_5 \geq 0$, so x_5 will leave the basis. The pivot equation comes from the second line above:

$$x_1 = \frac{1}{4}(4 - x_5 - 2x_2) = 1 - \frac{1}{4}x_5 - \frac{1}{2}x_2.$$

It replaces the second line in the dictionary and influences the others:

$$\begin{array}{l} x_4 = 2 + (1/4)x_5 - (5/2)x_2 + 2x_3 \\ x_1 = 1 - (1/4)x_5 - (1/2)x_2 \\ x_6 = 1 + (1/4)x_5 - (1/2)x_2 - x_3 \\ \hline f = 4 - x_5 + 2x_3 \end{array}$$

Now the entering variable is x_3 (check the bottom row) and the leaving variable is x_6; we pivot on the third equation in the dictionary, namely,

$$x_3 = 1 + \frac{1}{4}x_5 - \frac{1}{2}x_2 - x_6.$$

The new dictionary is

$$\begin{array}{rl} x_4 &= 4 + (3/4)x_5 - (7/2)x_2 - 2x_6 \\ x_1 &= 1 - (1/4)x_5 - (1/2)x_2 \\ x_3 &= 1 + (1/4)x_5 - (1/2)x_2 - x_6 \\ \hline f &= 6 - (1/2)x_5 - x_2 - 2x_6 \end{array}$$

There are no positive coefficients in the objective row, so this dictionary encodes an optimal solution. Setting the nonbasic variables x_2, x_5, and x_6 to 0 reveals the solution:

$$f_{\text{MAX}} = 6, \quad \text{attained at} \quad (x_1, x_2, x_3) = (1, 0, 1).$$

(b) The new objective $g = 3x_1 + 2x_2 + x_3$ can be expressed in terms of the variables basic in the final dictionary above by using the equations it supplies:

$$g = 3\left(1 - \frac{1}{4}x_5 - \frac{1}{2}x_2\right) + 2x_2 + \left(1 + \frac{1}{4}x_5 - \frac{1}{2}x_2 - x_6\right) = 4 - \frac{1}{2}x_5 - x_6.$$

Replacing the bottom line of the dictionary above with this one gives a feasible dictionary for the g-problem:

$$\begin{array}{rl} x_4 &= 4 + (3/4)x_5 - (7/2)x_2 - 2x_6 \\ x_1 &= 1 - (1/4)x_5 - (1/2)x_2 \\ x_3 &= 1 + (1/4)x_5 - (1/2)x_2 - x_6 \\ \hline g &= 4 - (1/2)x_5 \qquad\qquad\quad - x_6 \end{array}$$

This dictionary encodes a whole family of optimal solutions, because the objective row is unaffected by changes in the non-basic variable x_2. We can use any $x_2 \geq 0$ compatible with the nonnegativity constraints on the basic variables: the tightest of these is $x_4 \geq 0$, which requires $x_2 \leq 8/7$. In summary,

$$g_{\text{MAX}} = 4, \quad \text{attained at every point} \quad (x_1, x_2, x_3) = (1 - \tfrac{1}{2}x_2, x_2, 1 - \tfrac{1}{2}x_2) \text{ where } 0 \leq x_2 \leq \frac{8}{7}.$$

(The endpoints of the segment of maximizing points are $(1, 0, 1)$ and $(3/7, 8/7, 3/7)$.

(c) As in part (b), we first express the new objective in terms of the current basis:

$$h = 4x_1 + 4x_2 = 4\left(1 - \frac{1}{4}x_5 - \frac{1}{2}x_2\right) + 4x_2 = 4 - x_5 + 2x_2.$$

Replacing the objective row of the current dictionary with this one gives a feasible dictionary that is not optimal:

$$\begin{array}{rl} x_4 &= 4 + (3/4)x_5 - (7/2)x_2 - 2x_6 \\ x_1 &= 1 - (1/4)x_5 - (1/2)x_2 \\ x_3 &= 1 - (1/4)x_5 - (1/2)x_2 - x_6 \\ \hline h &= 4 - x_5 + 2x_2 \end{array}$$

Now x_2 enters the basis (check the objective row), and x_4 leaves (the tightest constraint on x_2 comes from the x_4-equation). So we pivot on the x_4-equation:

$$x_2 = \frac{2}{7}\left[4 + \frac{3}{4}x_5 - x_4 - 2x_6\right] = \frac{8}{7} + \frac{3}{14}x_5 - \frac{2}{7}x_4 - \frac{4}{7}x_6.$$

This leads to an optimal dictionary:

$$\begin{array}{rl} x_2 = & (8/7) + (3/14)x_5 - (2/7)x_4 - (4/7)x_6 \\ x_1 = & (3/7) - (5/14)x_5 + (1/7)x_4 + (2/7)x_6 \\ x_3 = & (3/7) + \ \ (1/7)x_5 + (1/7)x_4 - (5/7)x_6 \\ \hline h = & (44/7) - \ \ (4/7)x_5 - (4/7)x_4 - (8/7)x_6 \end{array}$$

There is a unique optimum:

$$h_{\mathrm{MAX}} = \frac{44}{7}, \quad \text{attained at} \quad (x_1, x_2, x_3) = \left(\frac{3}{7}, \frac{8}{7}, \frac{3}{7}\right).$$

3. Read textbook problem 1.6 about the meat packing plant. The story in the question setup is translated into a standard-form LP that appears on page 465. Look at that, too. Then ...

(a) Explain the meaning of each variable in the given LP formulation. Show that your explanation predicts a net profit of \$9965 for the sample schedule in the question statement.

(b) Use a computer to solve the given LP. Find the maximum profit (it should be larger than \$9965!) and present a table showing the schedule that achieves it. Also hand in a computer printout showing exactly what you typed into the solver and what it returned.

NOTES: Use any computer package you like—just say which one you choose. Possibilities:

1. Use LINDO, a special linear programming solver available on the machines in the Math/Stat undergraduate computer lab, room LSK 121. Access instructions should have arrived in your email last week. No special skills are required: just start LINDO, select "Help" from the menu, and try things.
2. Use the online simplex method tool cited on the course web page. The tool has an "example" button that shows you how it works, and another sample is posted on the course web page.

Most of the work in part (b) is in figuring out how to communicate your mathematical problem to the software, and how to interpret the results. These are skills worth learning now: once you have them, you can check your manual calculations and/or solve larger problems with ease.

(a) The LP formulated in the book says

$$\begin{array}{ll} \text{maximize} & f = 6x_1 + 3x_2 + 8x_3 + 3x_4 + 9x_5 + 5x_6 \\ \text{subject to} & x_1 + x_2 \le 480, \\ & x_3 + x_4 \le 400, \\ & x_5 + x_6 \le 230, \\ & x_1 + x_3 + x_5 \le 420, \\ & x_2 \quad x_4 \quad x_6 \le 250, \\ & x_1, x_2, x_3, x_4, x_5, x_6 \ge 0. \end{array}$$

The variables have the following meanings:

$$\begin{aligned} x_1 &= \text{number of hams smoked on regular time} \\ x_2 &= \text{number of hams smoked on overtime} \\ x_3 &= \text{number of bellies smoked on regular time} \\ x_4 &= \text{number of bellies smoked on overtime} \\ x_5 &= \text{number of picnics smoked on regular time} \\ x_6 &= \text{number of picnics smoked on overtime} \end{aligned}$$

The first three constraints encode the total supply of each type of meat every day. The next two constraints encode the total number of items that can be smoked on regular time (420) and on overtime (250). Notice that

$$\begin{aligned} \text{number of hams sold fresh} &= 480 - x_1 - x_2, \\ \text{number of bellies sold fresh} &= 400 - x_3 - x_4, \\ \text{number of picnics sold fresh} &= 230 - x_5 - x_6. \end{aligned}$$

The total net profit for a production plan $(x_1, \ldots, x_6)$ can be computed using the prices given in the problem formulation:

$$\begin{aligned} P = 8(480 - x_1 - x_2) + 14x_1 + 11x_2 \\ 4(400 - x_3 - x_4) + 12x_3 + 7x_4 \\ 4(230 - x_5 - x_6) + 13x_5 + 9x_6 \\ = 6x_1 + 3x_2 + 8x_3 + 3x_4 + 9x_5 + 5x_6 + 6360. \end{aligned}$$

The first line comes from selling three styles of ham, the second from three styles of bellies, and the third from three styles of picnics. The bottom line comes from algebra. The constant 5880 represents he net profit from selling all products fresh ($\mathbf{x} = \mathbf{0}$). The constant also spoils the linearity property of the total profit P for inputs $(x_1, \ldots, x_6)$, so we choose to maximize just the number we can add to 6360 by smoking some products: this extra profit equals

$$f = 6x_1 + 3x_2 + 8x_3 + 3x_4 + 9x_5 + 5x_6.$$

Choosing $\mathbf{x} = (280, 35, 70, 35, 70, 105)$ gives $f = 3605$. Therefore the total profit for this schedule is $P = 3605 + 6360 = 9965$, as asserted in the text.

(b) The online simplex solver returns the maximizing schedule

$$(x_1, \ldots, x_6) = (0, 40, 400, 0, 20, 210).$$

Here is the interpretation:

	Fresh	Smoked (reg)	Smoked (overtime)
Hams	440	0	40
Bellies	0	400	0
Picnics	0	20	210

The improvement on selling eveything fresh is $f_{\text{MAX}} = 4550$, so the total net profit with the optimal schedule is

$$P_{\text{MAX}} = \$6360 + \$4550 = \$10\,910.$$

4. Consider the following problem, in which a real parameter k appears:

$$\begin{array}{ll} \text{maximize} & f = -3x_1 + x_2 + kx_3 \\ \text{subject to} & x_1 - x_2 \le 0, \\ & -2x_1 + x_3 \le 1, \\ & -2x_2 + x_3 \le 2, \\ & x_1 + x_2 - x_3 \le 6, \\ & x_1, x_2, x_3 \ge 0. \end{array}$$

Notice that changing the value of k makes no difference to the feasible region.

(a) Suppose $k = 1$. Apply the Simplex Method and answer these questions:

(i) Is the problem bounded?

(ii) If the problem is bounded, find all maximizing points and their corresponding values.

(iii) If the problem is unbounded, find a feasible point whose f-value is larger than 10^{100}.

(b) Repeat part (a) with $k = 2$.

(a) When $k = 1$, the initial dictionary for the given problem is

$$D_0: \quad \begin{array}{l} f = -3x_1 + x_2 + x_3 \\ \hline x_4 = -x_1 + x_2 \\ x_5 = 1 + 2x_1 - x_3 \\ x_6 = 2 + 2x_2 - x_3 \\ x_7 = 6 - x_1 - x_2 + x_3 \end{array}$$

Choose x_2 (coefficient tied with x_3, select by Anstee's smallest subscript rule) to enter the basis and x_7 to leave: the pivot equation is $x_2 = 6 - x_1 - x_7 + x_3$, leading to

$$D_1: \quad \begin{array}{l} f = 6 - 4x_1 - x_7 + 2x_3 \\ \hline x_4 = 6 - 2x_1 - x_7 + x_3 \\ x_5 = 1 + 2x_1 - x_3 \\ x_6 = 14 - 2x_1 - 2x_7 + x_3 \\ x_2 = 6 - x_1 - x_7 + x_3 \end{array}$$

Now the entering variable is unambiguously x_3, and the leaving variable must be x_5. With pivot equation $x_3 = 1 + 2x_1 - x_5$, we get

$$D_2: \quad \begin{array}{l} f = 8 - x_7 - 2x_5 \\ \hline x_4 = 7 - x_7 - x_5 \\ x_3 = 1 + 2x_1 - x_5 \\ x_6 = 15 - 2x_7 - x_5 \\ x_2 = 7 + x_1 - x_7 - x_5 \end{array}$$

(i) Since there are no positive coefficients left in the objective row, this problem is bounded.

(ii) The coefficient of x_1 in the objective row is 0, indicating that the maximizing point is not unique. Any choice of $x_1 \geq 0$ is allowed, provided all equations in dictionary D_2 are respected. In particular, every point of the form

$$(x_1, x_2, x_3) = (x_1, 7 + x_1, 1 + 2x_1), \quad x_1 \geq 0$$

is feasible for the problem and gives the maximum value, $f_{\text{MAX}} = 8$.

(iii) The problem is bounded, so there is nothing to do here.

(b) Changing k to 2 affects only the objective function, not the constraints. So we can simply swap out the old objective row and swap in the new, taking care to respect dictionary D_2:

$$f = -3x_1 + x_2 + 2x_3 = -3x_1 + (7 + x_1 - x_7 - x_5) + 2(1 + 2x_1 - x_5) = 9 + 2x_1 - x_7 - 3x_5.$$

This leads to

$$\widetilde{D}_2 : \quad \begin{array}{rcl} f &=& 9 + 2x_1 - x_7 - 3x_5 \\ \hline x_4 &=& 7 - x_7 - x_5 \\ x_3 &=& 1 + 2x_1 - x_5 \\ x_6 &=& 15 - 2x_7 - x_5 \\ x_2 &=& 7 + x_1 - x_7 - x_5 \end{array}$$

(i) The positive coefficient on x_1 in the objective row indicates that increasing the value of x_1 will increase f. Earlier rows in the dictionary provide no limit on how much x_1 can increase, so this problem is unbounded.

(ii) This applies only for the bounded case, not here.

(iii) For any $x_1 \geq 0$, the input point $(x_1, x_2, x_3) = (x_1, 7 + x_1, 1 + 2x_1)$ is feasible for the problem and gives $f = 9 + 2x_1$. To get a value $f > 10^{100}$, we can choose $x_1 = 0.5 \times 10^{100}$: the corresponding input point is

$$(x_1, x_2, x_3) = (0.5 \times 10^{100},\ 7 + 0.5 \times 10^{100},\ 1 + 10^{100}).$$

(Any point of the same form, with $x_1 > 0.5 \times 10^{100} - 4.5$, will do the job.)

5. Consider a general feasible dictionary.

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j, \qquad i = 1, 2, \dots, m$$
$$z = v + \sum_{j=1}^{n} c_j x_j$$

in which $x_1, \dots x_n$ are non-basic and $x_{n+1}, \dots, x_{n+m}$ are basic. Prove that if we can choose an entering variable, but there are no leaving variables then the objective function can be made arbitrarily large and so the problem is unbounded.

The given dictionary is known to be feasible. Therefore

$$b_i \geq 0 \quad \text{for each } i = 1, 2, \dots, m.$$

Suppose the entering variable is x_k: this means that $k \in \{1, 2, \dots, j\}$ is a nonbasic subscript for which the corresponding coefficient obeys $c_k > 0$. We choose the leaving variable by selecting the tightest constraint in the following list:

$$0 \leq b_i - a_{ik}x_k, \quad \text{i.e.,} \quad a_{ik}x_k \leq b_i, \qquad i = 1, 2, \dots, m.$$

The process breaks down if and only if the list has no tight constraints at all, i.e., if

$$a_{ik} \leq 0 \quad \text{for each} \quad i = 1, 2, \dots, m. \tag{$**$}$$

Suppose this happens. Look at the solution of the given dictionary in which all nonbasic variables *except* x_k are stuck at 0:

$$\begin{aligned} x_{n+1} &= b_1 - a_{1k}x_k, \\ x_{n+2} &= b_2 - a_{2k}x_k, \\ &\vdots \\ x_{n+m} &= b_m - a_{mk}x_k, \\ \hline z &= v + c_k x_k. \end{aligned}$$

Line $(**)$ guarantees that the values of the basic variables $x_{n+1}, \dots, x_{n+m}$ will be nonnegative for any value of $x_k \geq 0$, no matter how large. So the point below is a feasible solution for arbitrary $x_k > 0$:

$$(0, \dots, 0, x_k, 0, \dots, 0, b_1 - a_{1k}x_k, b_2 - a_{2k}x_k, \dots, b_{n+m} - a_{(n+m)k}x_k).$$

The value of this point is shown above: $z = v + c_k x_k$. Since $c_k > 0$, and there is no limit on how large $x_k > 0$ can be, the objective value z can be made arbitrarily large. That's exactly what it means to say the problem is unbounded.

6. Find all maximizing points (if any) and their objective value:

$$\begin{array}{ll} \text{maximize} & f = 3x_1 + x_2 \\ \text{subject to} & x_1 + 2x_2 \le 5, \\ & x_1 + x_2 - x_3 \le 2, \\ & 7x_1 + 3x_2 - 5x_3 \le 20, \\ & x_1, x_2, x_3 \ge 0. \end{array}$$

The initial dictionary for this problem is

$$D_0: \quad \begin{array}{l} g = 3x_1 + x_2 \\ \hline x_4 = 5 - x_1 - 2x_2 \\ x_5 = 2 - x_1 - x_2 + x_3 \\ x_6 = 20 - 7x_1 - 3x_2 + 5x_3 \end{array}$$

Choose x_1 to enter the basis and x_5 to leave, with pivot equation $x_1 = 2 - x_5 - x_2 + x_3$:

$$D_1: \quad \begin{array}{l} g = 6 - 3x_5 - 2x_2 + 3x_3. \\ \hline x_4 = 3 + x_5 - x_2 - x_3 \\ x_1 = 2 - x_5 - x_2 + x_3 \\ x_6 = 6 + 7x_5 + 4x_2 - 2x_3 \end{array}$$

Choose x_3 to enter the basis and x_4 to leave (using the smaller subscript to break the tie between leaving vars). The pivot equation is $x_3 = 3 + x_5 - x_2 - x_4$, and it leads to

$$D_2: \quad \begin{array}{l} g = 15 - 5x_2 - 3x_4 \\ \hline x_3 = 3 + x_5 - x_2 - x_4 \\ x_1 = 5 - 2x_2 - x_4 \\ x_6 = 5x_5 + 6x_2 + 2x_4 \end{array}$$

This is an optimal dictionary that allows nonuniqueness: choosing $x_2 = 0$, $x_4 = 0$, and any $x_5 \ge 0$ will produce an objective value of $g_{\text{MAX}} = 15$. There are no limits on the size of x_5, so we have a whole ray of maximizing points. In terms of the original choice variables, the ray has the parametric form

$$(x_1, x_2, x_3) = (5, 0, 3 + x_5), \qquad x_5 \ge 0.$$

7. Consider this problem involving a real parameter k:

$$\begin{aligned} \text{Maximize } f = \;& x_1 + x_2 + kx_3 \\ \text{Subject to } \;& x_1 \le 2 \\ & x_2 \le 2 \\ & x_3 \le 4 \\ & 4x_1 + 4x_2 + x_3 \le 16 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

(a) Use the simplex method to solve the problem when $k = -1$. Identify all maximizing points.

(b) Find all k-values for which this problem has a unique maximizing point. Identify the maximizer.

(c) Solve the problem when $k = 1$. Identify all maximizing points.

(d) Find all k for which the set of maximizers in the problem is a line segment in (x_1, x_2, x_3)-space. Identify the endpoints of the segment.

(e) Find all k (if any) for which the set of maximizers includes 3 or more different basic feasible solutions. Identify those points.

(a) Introduce slack variables named s_1, s_2, s_3, s_4 to obtain this feasible dictionary when $k = -1$:

$$\begin{array}{rl} \zeta = & \quad\;\; x_1 + x_2 - x_3 \\ \hline s_1 = & 2 - x_1 \\ s_2 = & 2 \qquad\;\; - x_2 \\ s_3 = & 4 \qquad\qquad\quad - x_3 \\ s_4 = & 16 - 4x_1 - 4x_2 - x_3 \end{array}$$

This dictionary is improvable: x_1 will enter, s_1 will leave. Result:

$$\begin{array}{rl} \zeta = & 2 + x_2 - x_3 - s_1 \\ \hline x_1 = & 2 \qquad\qquad\quad - s_1 \\ s_2 = & 2 - x_2 \\ s_3 = & 4 \qquad - x_3 \\ s_4 = & 8 - 4x_2 - x_3 + 4s_1 \end{array}$$

This dictionary is improvable: x_2 will enter, s_2 will leave. We get

$$\begin{array}{rl} \zeta = & 4 - x_3 - s_1 - s_2 \\ \hline x_1 = & 2 \qquad\; - s_1 \\ x_2 = & 2 \qquad\qquad\quad - s_2 \\ s_3 = & 4 - x_3 \\ s_4 = & \quad\; - x_3 + 4s_1 + 4s_2 \end{array}$$

This dictionary is optimal. In terms of the original variables only, we have

$$\zeta_{\max} = 4, \qquad \text{attained uniquely at} \quad (x_1, x_2, x_3) = (2, 2, 0).$$

(b) Plugging $\zeta = x_1 + x_2 + kx_3$ into the optimal dictionary in part (a) changes only the objective row, which becomes

$$\zeta = (2 - s_1) + (2 - s_2) + kx_3 = 4 + kx_3 - s_1 - s_2.$$

Of course this agrees with the result in part (a) in the special case where $k = -1$, but now it reveals that the point found in (a) actually gives the unique minimizer whenever $k < 0$. If $k = 0$, increasing x_3 will not subtract from the objective value. However, the final row in the dictionary above shows that x_3 cannot increase without breaking the feasibility condition $s_4 \geq 0$. So the point found in (a) continues to be the unique maximizer when $k = 0$. What if $k > 0$? Then the final dictionary from (a) is improvable: x_3 should enter the basis and s_4 must leave. This (degenerate) pivot leads to

$$\begin{array}{l} \zeta = 4 - ks_4 - (1-4k)s_1 - (1-4k)s_2 \\ \hline x_1 = 2 \qquad - \quad s_1 \\ x_2 = 2 \qquad\qquad - \quad s_2 \\ s_3 = 4 + s_4 - 4s_1 - 4s_2 \\ x_3 = \quad - s_4 + 4s_1 + 4s_2 \end{array}$$

The nonbasic coefficients in the objective row are all negative whenever $0 < k < 1/4$, so in this interval we also have the unique maximizer found in (a). But when $k = 1/4$, the coefficients of both s_1 and s_2 in the objective become 0 and this allows nonuniqueness. Looking ahead to parts (c)–(e), we conclude that the problem has a unique maximizer if and only if $k < 1/4$; for all such k, the maximizing point is $(2, 2, 0)$.

(c) Substituting $k = 1$ into the dictionary from (b) gives

$$\begin{array}{l} \zeta = 4 - s_4 + 3s_1 + 3s_2 \\ \hline x_1 = 2 \qquad - s_1 \\ x_2 = 2 \qquad\qquad - s_2 \\ s_3 = 4 + s_4 - 4s_1 - 4s_2 \\ x_3 = \quad - s_4 + 4s_1 + 4s_2 \end{array}$$

Pivoting s_1 into the basis will improve this; the leaving variable is s_3, and the updated dictionary is

$$\begin{array}{l} \zeta = 7 - (1/4)s_4 - (3/4)s_3 \\ \hline x_1 = 1 - (1/4)s_4 + (1/4)s_3 + s_2 \\ x_2 = 2 \qquad\qquad\qquad\qquad - s_2 \\ s_1 = 1 + (1/4)s_4 - (1/4)s_3 - s_2 \\ x_3 = 4 \qquad - \qquad s_3 \end{array}$$

This dictionary is optimal: it shows that $\zeta_{\max} = 7$, and that the choices $s_4 = 0$ and $s_3 = 0$ are essential to achieve it (because these nonbasic variables have negative objective coefficients). But the choice of s_2 is more flexible, because it has a zero coefficient in the objective row. We can use any $s_2 \geq 0$ compatible with the nonnegativity constraints on the basic variables. Of these, the most restrictive comes from s_1, and it shows that the interval of permissible s_2-values is $[0, 1]$. In parametric form, showing original variables only, we have maximizers at

$$(x_1, x_2, x_3) = (1 + s_2, 2 - s_2, 4), \qquad 0 \leq s_2 \leq 1.$$

This describes a segment with endpoints at $(1,2,4)$ and $(2,1,4)$.

(d) Plugging $\zeta = x_1 + x_2 + kx_3$ into the optimal dictionary in part (c) changes only the objective row, which becomes

$$\zeta = (3+4k) - (1/4)s_4 - (k-1/4)s_3.$$

The dictionary remains optimal for each $k \geq 1/4$. When $k > 1/4$, the discussion in (c) applies without change: there is a tie for the maximum value at every point on the segment joining $(1,2,4)$ to $(2,1,4)$. Looking ahead to part (e), we find that *the problem has a line segment of maximizers if and only if $k > 1/4$.*

(e) What happens when $k = 1/4$? Substitution into the dictionary from (b) gives the dictionary

$$\begin{array}{rl} \zeta = & 4 - (1/4)s_4 \\ \hline x_1 = & 2 \qquad\qquad - s_1 \\ x_2 = & 2 \qquad\qquad\qquad - s_2 \\ s_3 = & 4 + \quad s_4 - 4s_1 - 4s_2 \\ x_3 = & \quad - \quad s_4 + 4s_1 + 4s_2 \end{array}$$

This is an optimal dictionary: $\zeta_{\max} = 4$. Every maximizing point must have $s_4 = 0$. However, two nonbasic variables are allowed to vary without compromising the objective value. Choosing $s_1 = 0$ and $s_2 = 0$ gives a basic maximizer where $(x_1, x_2, x_3) = (2,2,0)$. Choosing $s_1 = 1$ and $s_2 = 0$ gives a basic maximizer at $(x_1, x_2, x_3) = (1,2,4)$. Choosing $s_1 = 0$ and $s_2 = 1$ gives a basic maximizer at $(x_1, x_2, x_3) = (2,1,4)$. All points on the flying triangle in $\mathbb{R}^3$ with these three corners are tied for the maximum when $k = 1/4$.