M340(921) Solutions—Problem Set 1

1. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 6 & 0 & 0 & 3 & -1 & 0 \\ 1 & 3 & 0 & 1 & 9 & 2 & -1 \\ 1 & 3 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}.$$

Which of the five vectors below, if any, are "basic solutions" of $A\mathbf{x} = \mathbf{b}$? Explain your decision for each vector by making reference to the definition presented in class.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 30 \\ 0 \\ -10 \\ 5 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 0 \\ 5 \\ -135 \\ 10 \\ 20 \\ -10 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We are working on the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 6 & 0 & 0 & 3 & -1 & 0 \\ 1 & 3 & 0 & 1 & 9 & 2 & -1 \\ 1 & 3 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}.$$

(i) The vector $\mathbf{x} = (0, 0, 0, 0, 3, 2, 1)$ **is not** a basic solution because it does not satisfy the equation. In fact, $A\mathbf{x} = (7, 30, 4)$.

(ii) The vector $\mathbf{x} = (0, 0, 0, 30, 0, -10, 5)$ **is** a basic solution because it satisfies $A\mathbf{x} = \mathbf{b}$ and its nonzero entries correspond to three linearly independent columns of A. In more detail, choosing $\mathcal{B} = \{4, 6, 7\}$ leads to a complementary set $\mathcal{N} = \{1, 2, 3, 5\}$ for which both defining requirements are satisfied, namely,

$$x_j = 0 \qquad \forall j \in \mathcal{N} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{is invertible.}$$

(iii) The vector $\mathbf{x} = (0, 0, 5, -135, 10, 20, -10)$ **is not** a basic solution because it has too many nonzero components. Since the matrix A has just three rows, every basic solution can have at most three nonzero entries.

(iv) The vector $\mathbf{x} = (2, 1, 0, 0, 0, 0, 0)$ **is not** a basic solution because its nonzero components select columns from A that are linearly dependent.

(v) The vector $\mathbf{x} = (5, 0, 0, 0, 0, 0, 0)$ **is** a basic solution because it satisfies $A\mathbf{x} = \mathbf{b}$ and there is a compatible choice of basis. In fact there are several different bases that work: any collection of three linearly independent columns from A that includes the first column will serve. One specific choice involves the basic index set $\mathcal{B} = \{1, 3, 4\}$, which induces $\mathcal{N} = \{2, 5, 6, 7\}$. Clearly we have both defining properties, namely,

$$x_j = 0 \qquad \forall j \in \mathcal{N} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{is invertible.}$$

2. Consider the matrices B and C below, in which there are parameters β and γ:

$$B = \begin{bmatrix} \beta & 1 & 1 \\ 1 & \beta & 1 \\ 1 & 1 & \beta \end{bmatrix}, \qquad C = \begin{bmatrix} \gamma & -1 & -1 \\ -1 & \gamma & -1 \\ -1 & -1 & \gamma \end{bmatrix}.$$

(a) The matrix product BC produces a scalar multiple of I, the identity matrix, if and only if a certain equation relating β to γ is satisfied. Find this equation.

(b) Use your result from (a) to find a formula for B^{-1}. State the restrictions on β required to make your formula correct.

(c) For each of the β-values where your formula in (b) breaks down, find a *nonzero* vector $\mathbf{v}$ such that $B\mathbf{v} = \mathbf{0}$. [*Remark*: There are many such vectors; any one of them is acceptable. Producing such a vector provides independent verification that B is not invertible.]

(d) Find B^{-1}, given

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

(a) The product matrix BC contains many copies of the same two expressions:

$$BC = \begin{bmatrix} \beta & 1 & 1 \\ 1 & \beta & 1 \\ 1 & 1 & \beta \end{bmatrix} \begin{bmatrix} \gamma & -1 & -1 \\ -1 & \gamma & -1 \\ -1 & -1 & \gamma \end{bmatrix} = \begin{bmatrix} p & s & s \\ s & p & s \\ s & s & p \end{bmatrix},$$

where $p = \beta\gamma - 2$ and $s = \gamma - b - 1$. We will have $BC = pI$ if and only if $s = 0$, i.e., $\gamma = \beta + 1$.

(b) When $\gamma = \beta + 1$ as in part (a), we have $p = \beta\gamma - 2 = \beta^2 + \beta - 2 = (\beta - 1)(\beta + 2)$ In this case, $BC = pI$, so

$$B\left(\frac{1}{p}C\right) = I \implies B^{-1} = \frac{1}{p}C = \frac{1}{(\beta+2)(\beta-1)} \begin{bmatrix} (\beta+1) & -1 & -1 \\ -1 & (\beta+1) & -1 \\ -1 & -1 & (\beta+1) \end{bmatrix}.$$

This formula is valid if and only if $\beta \notin \{-2, 1\}$.

(c) When $\beta = -2$, we have

$$B = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix},$$

and it's clear that $B\mathbf{v} = \mathbf{0}$ holds for $\mathbf{v} = (1, 1, 1)$. Indeed, row-reducing B shows

$$B \sim \begin{bmatrix} -1 & 1/2 & 1/2 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & -3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $B\mathbf{v} = \mathbf{0}$ holds if and only if $v_3 = 1$ and $v_2 = v_1$. The set of solutions is precisely the set of all scalar multiples of $(1, 1, 1)$.

When $\beta = 1$, we have

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and it's clear that $B\mathbf{v} = \mathbf{0}$ holds if and only if $v_1 + v_2 + v_3 = 0$. This equation has many solutions, including $(1, 1, -2)$.

(d) Inspired by the pattern above, let's investigate a matrix of this form:

$$C = \begin{bmatrix} \gamma & -1 & -1 & -1 & -1 \\ -1 & \gamma & -1 & -1 & -1 \\ -1 & -1 & \gamma & -1 & -1 \\ -1 & -1 & -1 & \gamma & -1 \\ -1 & -1 & -1 & -1 & \gamma \end{bmatrix}.$$

This makes a simple matrix product with B:

$$BC = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \gamma & -1 & -1 & -1 & -1 \\ -1 & \gamma & -1 & -1 & -1 \\ -1 & -1 & \gamma & -1 & -1 \\ -1 & -1 & -1 & \gamma & -1 \\ -1 & -1 & -1 & -1 & \gamma \end{bmatrix} = \begin{bmatrix} p & s & s & s & s \\ s & p & s & s & s \\ s & s & p & s & s \\ s & s & s & p & s \\ s & s & s & s & p \end{bmatrix},$$

where $p = 2\gamma - 4$ and $s = \gamma - 5$. To get a multiple of the identity on the right, we choose $\gamma = 5$. This gives $p = 6$, so $BC = 6I$, and

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}^{-1} = \frac{1}{6}C = \frac{1}{6} \begin{bmatrix} 5 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & -1 & 5 \end{bmatrix}.$$

3. Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x} \in \mathbb{R}^5$ and

$$A = \begin{bmatrix} 2 & 1 & 4 & 1 & 8 \\ 1 & 2 & -2 & 1 & 0 \\ 1 & 1 & 12 & 2 & 14 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -8 \end{bmatrix}.$$

Build a dictionary representation of this system by following these steps.

(a) Expand the matrix-vector notation $A\mathbf{x} = \mathbf{b}$ into three linear equations. In each equation, put the terms involving x_1, x_2, and x_4 on the left side and all the other terms on the right.

(b) Find the matrices B and N for which the system in (a) has the form

$$B\mathbf{x}_{\mathcal{B}} = \mathbf{b} - N\mathbf{x}_{\mathcal{N}},$$

where $\mathbf{x}_{\mathcal{B}} = (x_1, x_2, x_4)$ and $\mathbf{x}_{\mathcal{N}} = (x_3, x_5)$.

(c) Multiply through by B^{-1} to get a formula for $\mathbf{x}_{\mathcal{B}}$, then expand that back into dictionary notation. (To find B^{-1}, look through other problems on this assignment.)

(a) Expanding the matrix product $A\mathbf{x}$ leads to the following presentations of $A\mathbf{x} = \mathbf{b}$:

$$\left.\begin{array}{rl} 2x_1 + x_2 + 4x_3 + x_4 + 8x_5 = & 4 \\ x_1 + 2x_2 - 2x_3 + x_4 \qquad = & 0 \\ x_1 + x_2 + 12x_3 + 2x_4 + 14x_5 = & -8 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{rl} 2x_1 + x_2 + x_4 = & 4 - 4x_3 - 8x_5 \\ x_1 + 2x_2 + x_4 = & 2x_3 \\ x_1 + x_2 + 2x_4 = & -8 - 12x_3 - 14x_5. \end{array}\right.$$

(b) We can repackaging the second system above in matrix form as follows:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -8 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ -2 & 0 \\ 12 & 14 \end{bmatrix}.$$

This has the form $B\mathbf{x}_{\mathcal{B}} = \mathbf{b} - N\mathbf{x}_{\mathcal{N}}$, with

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad N = \begin{bmatrix} 4 & 8 \\ -2 & 0 \\ 12 & 14 \end{bmatrix}.$$

(c) Since B is invertible, $\quad B\mathbf{x}_{\mathcal{B}} = \mathbf{b} - N\mathbf{x}_{\mathcal{N}} \implies \mathbf{x}_{\mathcal{B}} = B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_{\mathcal{N}}$.
Thanks to a formula for B^{-1} in an earlier question, this gives the matrix equation

$$\mathbf{x}_{\mathcal{B}} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -8 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ -2 & 0 \\ 12 & 14 \end{bmatrix} \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \\ -7 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 5 \\ -11 & -11 \\ 17 & 17 \end{bmatrix} \begin{bmatrix} x_3 \\ x_5 \end{bmatrix}$$

Expanding this gives a set of equations in dictionary format:

$$\begin{aligned} x_1 &= 5 - (1/2)x_3 - (5/2)x_5 \\ x_2 &= 1 + (11/2)x_3 + (11/2)x_5 \\ x_4 &= -7 - (17/2)x_3 - (17/2)x_5. \end{aligned}$$

(The point: if B^{-1} is available, it's easy to build the corresponding dictionary.)

4. A certain system of 3 equations in 6 variables has this dictionary representation:

$$\begin{aligned} x_4 &= 1 + 2x_1 + \ \ x_2 - \ \ x_3 \\ x_5 &= 1 \qquad\quad + 2x_2 + \ \ x_3 \\ x_6 &= 1 + 5x_1 + 2x_2 - 3x_3 \end{aligned} \tag{$*$}$$

(a) Use some sequence of pivots to produce an equivalent dictionary in which the basic variables are $\{x_1, x_2, x_3\}$.

(b) Find M^{-1} for the matrix M defined below:

$$M = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}.$$

Suggestion: Notice that M is the matrix of coefficients that arises when the given system is presented in the form $z = Mx + b$ for suitable definitions of b, x, and z in $\mathbb{R}^3$.

(c) Suppose the column of constants in $(*)$ is changed from $(1,1,1)$ to (b_1, b_2, b_3). Find the basic solution of $(*)$ corresponding to the choice of $\{x_1, x_2, x_3\}$ as basic variables.
Hint: A quick and easy solution can be based on the result in (b).

(a) We start with the given dictionary representation:

$$\left[\begin{aligned} x_4 &= 1 + 2x_1 + \ \ x_2 - \ \ x_3 \\ x_5 &= 1 \qquad\quad + 2x_2 + \ \ x_3 \\ x_6 &= 1 + 5x_1 + 2x_2 - 3x_3 \end{aligned}\right.$$

To pivot x_3 into the basis and x_4 out, we rearrange the first equation above as follows:

$$x_3 = 1 + 2x_1 + x_2 - x_4$$

Using this to reduce the other equations gives

$$\left[\begin{aligned} x_3 &= \quad 1 + 2x_1 + \ \ x_2 - \ \ x_4 \\ x_5 &= \quad 2 + 2x_1 + 3x_2 - \ \ x_4 \\ x_6 &= -2 - \ \ x_1 - \ \ x_2 + 3x_4 \end{aligned}\right.$$

Next (avoiding fractions) we pivot x_1 into the basis and x_6 out. The last equation above gives the pivot equation:

$$x_1 = -2 - x_6 - x_2 + 3x_4$$

Substituting this into the other equations produces

$$\left[\begin{aligned} x_3 &= -3 - 2x_6 - x_2 + 5x_4 \\ x_5 &= -2 - 2x_6 + x_2 + 5x_4 \\ x_1 &= -2 - \ \ x_6 - x_2 + 3x_4 \end{aligned}\right.$$

Finally we pivot x_2 into the basis and x_5 out. The middle equation above gives the pivot equation:

$$x_2 = 2 + 2x_6 + x_5 - 5x_4$$

Then we substitute into the other equations, giving

$$\left[\begin{array}{rl} x_3 = & -5 - 4x_6 - x_5 + 10x_4 \\ x_2 = & 2 + 2x_6 + x_5 - 5x_4 \\ x_1 = & -4 - 3x_6 - x_5 + 8x_4 \end{array}\right.$$

(b) The given dictionary encodes the identity $z = Mx + b$ for the vector variables $z = (x_4, x_5, x_6)$ and $x = (x_1, x_2, x_3)$, in terms of the constant ingredients

$$M = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The transformed dictionary expresses $x = M^{-1}z - M^{-1}b$. An equivalent presentation with variables in the right order gives

$$\left[\begin{array}{rl} x_1 = & -4 + 8x_4 - x_5 - 3x_6 \\ x_2 = & 2 - 5x_4 + x_5 + 2x_6 \\ x_3 = & -5 + 10x_4 - x_5 - 4x_6 \end{array}\right.$$

Therefore

$$M = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \iff M^{-1} = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}.$$

(c) The identity $x = M^{-1}z - M^{-1}b$ with the definitions in (b) shows that the desired basic solution is obtained by setting $z = 0$, so $x = -M^{-1}b$. The result is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -8b_1 + b_2 + 3b_3 \\ 5b_1 - b_2 - 2b_3 \\ -10b_1 + b_2 + 4b_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

5. Consider the following system of linear equations:

$$\begin{aligned} 5x_1 - 4x_2 - x_3 + w_1 \quad\quad &= 10, \\ x_1 - \ x_2 \quad\quad + w_2 \quad &= \ 4, \\ -3x_1 + 4x_2 + x_3 \quad\quad + w_3 &= \ 1. \end{aligned}$$

(a) Find the basic solution (a 6-element vector) corresponding to the basic variables x_1, x_2, w_3.

(b) Express the given system in dictionary form, choosing $\{w_1, w_2, w_3\}$ as the set of basic variables.

(c) Express the system in dictionary form, choosing variables w_1, x_1, w_3 as basic. (*Hint*: Pivot.)

(d) Find all dictionary representations in which two of the three basic variables are w_1 and w_3. (Note that two of the four choices you need to think about are covered in parts (a)–(b). Remember that when solving systems of linear equations, existence of a solution is not always guaranteed!)

(a) Setting all non-basic variables to 0 leads to a 3×3 system to solve for the basic ones.

$$\begin{aligned} 5x_1 - 4x_2 \quad\quad &= 10, \\ x_1 - \ x_2 \quad\quad &= \ 4, \\ -3x_1 + 4x_2 + w_3 &= \ 1. \end{aligned}$$

The second equation gives $x_1 = 4 + x_2$; plugging this into the first equation gives

$$20 + x_2 = 10, \quad \text{i.e.,} \quad x_2 = -10.$$

Then the second equation gives $x_1 = -6$, and the third gives $w_3 = 23$. Answer:

$$(x_1, x_2, x_3, w_1, w_2, w_3) = (-6, -10, 0, 0, 0, 23).$$

(b) Solving the given system for w_1, w_2, w_3 in terms of the remaining variables is as simple as rearranging each equation independently. Answer:

$$\begin{aligned} w_1 &= 10 - 5x_1 + 4x_2 + x_3 \\ w_2 &= \ 4 - \ x_1 + \ x_2 \\ w_3 &= \ 1 + 3x_1 - 4x_2 - x_3 \end{aligned}$$

(c) To get the desired basis, we start with the basis in (b) and force w_2 to become nonbasic ("leave") and x_1 to become basic ("enter"). This requires putting x_1 on the left side of some equation and w_2 on the right. The second equation in the dictionary of (b) helps with this:

$$w_2 = 4 - x_1 + x_2 \qquad \Longleftrightarrow \qquad x_1 = 4 - w_2 + x_2.$$

Substituting the latter form into the other equations in the dictionary from (b) gives the desired new dictionary:

$$\begin{aligned} w_1 &= -10 + 5w_2 - x_2 + x_3 \\ x_1 &= \quad 4 - \ w_2 + x_2 \\ w_3 &= \quad 13 - 3w_2 - x_2 - x_3 \end{aligned}$$

(These three rows can appear in any order. On the right sides, the variables can appear in any order.)

(d) Given that two of the three basic variables must be w_1 and w_3, we have four choices for the remaining one.

(i) Basic variables $\{w_1, w_2, w_3\}$. This is covered in part (b).

(ii) Basic variables $\{w_1, x_1, w_3\}$. This is done in part (c)

(iii) Basic variables $\{w_1, x_2, w_3\}$. We can get this from the dictionary in (c) by pivoting x_2 into the basis and x_1 out. We do this by rearranging the second equation above, obtaining the "pivot equation"

$$x_2 = -4 + x_1 + w_2.$$

Using this in the remaining two equations produces the desired dictionary:

$$\begin{aligned} x_2 &= -4 + x_1 \qquad + w_2 \\ w_1 &= -6 - x_1 + x_3 + 4w_2 \\ w_3 &= 17 - x_1 - x_3 - 4w_2 \end{aligned}$$

(iv) Basic variables $\{w_1, x_3, w_3\}$. This is impossible. The given system of equations is equivalent to

$$\begin{aligned} -x_3 + w_1 \qquad &= 10 - 5x_1 + 4x_2 \\ 0 \qquad\qquad &= 4 - x_1 + x_2 - w_2 \\ x_3 \qquad + w_3 &= 1 + 3x_1 - 4x_2 \end{aligned}$$

It is impossible to solve uniquely for the variables on the left in terms of the ones on the right. One way to justify this is to notice that the left side values can be stacked into the vector

$$\begin{bmatrix} -x_3 + w_1 \\ 0 \\ x_3 \qquad + w_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ w_1 \\ w_3 \end{bmatrix},$$

and the coefficient matrix here is non-invertible.

In summary, the general shape of this system makes it look like there could be 4 possible choices for a basis involving w_1 and w_3. However, the particulars of the coefficients in this problem mean that only 3 of these choices are acceptable.