Math 340 Problem Set 1

Due in class on Friday 17 May 2013

Please review the notes on academic integrity on the last page. *These apply throughout the course.*

1. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 6 & 0 & 0 & 3 & -1 & 0 \\ 1 & 3 & 0 & 1 & 9 & 2 & -1 \\ 1 & 3 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}.$$

Which of the five vectors below, if any, are "basic solutions" of $A\mathbf{x} = \mathbf{b}$? Explain your decision for each vector by making reference to the definition presented in class.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 30 \\ 0 \\ -10 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 5 \\ -135 \\ 10 \\ 20 \\ -10 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

2. Consider the matrices B and C below, in which there are parameters β and γ:

$$B = \begin{bmatrix} \beta & 1 & 1 \\ 1 & \beta & 1 \\ 1 & 1 & \beta \end{bmatrix}, \qquad C = \begin{bmatrix} \gamma & -1 & -1 \\ -1 & \gamma & -1 \\ -1 & -1 & \gamma \end{bmatrix}.$$

(a) The matrix product BC produces a scalar multiple of I, the identity matrix, if and only if a certain equation relating β to γ is satisfied. Find this equation.

(b) Use your result from (a) to find a formula for B^{-1}. State the restrictions on β required to make your formula correct.

(c) For each of the β-values where your formula in (b) breaks down, find a *nonzero* vector $\mathbf{v}$ such that $B\mathbf{v} = \mathbf{0}$. [*Remark*: There are many such vectors; any one of them is acceptable. Producing such a vector provides independent verification that B is not invertible.]

(d) Find B^{-1}, given

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

3. Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x} \in \mathbb{R}^5$ and

$$A = \begin{bmatrix} 2 & 1 & 4 & 1 & 8 \\ 1 & 2 & -2 & 1 & 0 \\ 1 & 1 & 12 & 2 & 14 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -8 \end{bmatrix}.$$

More . . .

Build a dictionary representation of this system by following these steps.

(a) Expand the matrix-vector notation $A\mathbf{x} = \mathbf{b}$ into three linear equations. In each equation, put the terms involving x_1, x_2, and x_4 on the left side and all the other terms on the right.

(b) Find the matrices B and N for which the system in (a) has the form

$$B\mathbf{x}_{\mathcal{B}} = \mathbf{b} - N\mathbf{x}_{\mathcal{N}},$$

where $\mathbf{x}_{\mathcal{B}} = (x_1, x_2, x_4)$ and $\mathbf{x}_{\mathcal{N}} = (x_3, x_5)$.

(c) Multiply through by B^{-1} to get a formula for $\mathbf{x}_{\mathcal{B}}$, then expand that back into dictionary notation. (To find B^{-1}, look through other problems on this assignment.)

4. A certain system of 3 equations in 6 variables has this dictionary representation:

$$\begin{aligned} x_4 &= 1 + 2x_1 + x_2 - x_3 \\ x_5 &= 1 + 2x_2 + x_3 \\ x_6 &= 1 + 5x_1 + 2x_2 - 3x_3 \end{aligned} \qquad (*)$$

(a) Use some sequence of pivots to produce an equivalent dictionary in which the basic variables are $\{x_1, x_2, x_3\}$.

(b) Find M^{-1} for the matrix M defined below:

$$M = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}.$$

Suggestion: Notice that M is the matrix of coefficients that arises when the given system is presented in the form $z = Mx + b$ for suitable definitions of b, x, and z in $\mathbb{R}^3$.

(c) Suppose the column of constants in $(*)$ is changed from $(1, 1, 1)$ to (b_1, b_2, b_3). Find the basic solution of $(*)$ corresponding to the choice of $\{x_1, x_2, x_3\}$ as basic variables.
Hint: A quick and easy solution can be based on the result in (b).

5. Consider the following system of linear equations:

$$\begin{aligned} 5x_1 - 4x_2 - x_3 + w_1 &= 10, \\ x_1 - x_2 + w_2 &= 4, \\ -3x_1 + 4x_2 + x_3 + w_3 &= 1. \end{aligned}$$

(a) Find the basic solution (a 6-element vector) corresponding to the basic variables x_1, x_2, w_3.

(b) Express the given system in dictionary form, choosing $\{w_1, w_2, w_3\}$ as the set of basic variables.

(c) Express the system in dictionary form, choosing variables w_1, x_1, w_3 as basic. (*Hint*: Pivot.)

(d) Find all dictionary representations in which two of the three basic variables are w_1 and w_3. (Note that two of the four choices you need to think about are covered in parts (a)–(b). Remember that when solving systems of linear equations, existence of a solution is not always guaranteed!)

6. Read Chapters 1 and 2 of the textbook. There is nothing to hand in . . . just do it. (The sooner the better!)

Working together and academic integrity

Original at http://www.math.ubc.ca/~andrewr/maths340/integrity.html

- ⋆ We have no objection in principle to collaboration on the homework, provided that it is done in a way that maximizes the benefit of the homework to all people involved.
- ⋆ However, if one person simply tells another how to do a problem then it completely defeats the purpose of having homework problems.
- ⋆ You get maximum benefit from a homework problem if you work hard on it alone before combining your ideas with those of someone else.
- ⋆ The work that you turn in with your name on it should represent your own solutions, written in your own words, regardless of whether you arrived at some of those solutions in collaboration with others.
- ⋆ You may not simply copy someone else's homework and turn it in as your own. This violates UBC's Academic Integrity Code.
- ⋆ Copying solutions that you find on the web or elsewhere is a similar violation, and will receive similar treatment.
- ⋆ We take academic integrity very seriously and will follow University procedures in all cases of suspected cheating. Students found cheating face severe penalties, ranging all the way up to expulsion from the University.