Name______________________________ **Signature** ______________________________

UBC Student Number____________________

The University of British Columbia

Final Examination – 27 June 2013

Mathematics 340

Linear Programming

Closed book examination

Time: 150 minutes

Special Instructions:

To receive full credit, all answers must be supported with clear and correct derivations. No calculators, notes, or other aids are allowed.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		15
2		15
3		15
4		15
5		15
6		20
7		5
Total		100

[15] **1.** (a) Use a two-phase simplex method with Anstee's pivoting rules to solve the problem below.

$$
\begin{aligned}
\text{maximize} \quad & f = 2x_1 - x_2 + x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + x_3 \geq 4 \\
& 2x_2 - x_3 \geq 3 \\
& x_1 + x_3 \leq 2 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

[Notes: Look carefully at the inequalities.There is more space to work on the next page.]

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(b) Suppose the objective function in the problem from part (a) is changed to

$$f = c_1x_1 + c_2x_2 + x_3.$$

Find all pairs (c_1, c_2) for which the basis identified in your solution for part (a) remains optimal. Sketch the set of all such pairs on a Cartesian plane with axes labelled c_1 and c_2; locate the special point $(c_1, c_2) = (2, -1)$ associated with part (a) on your sketch.

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[15] **2.** Given $h(x_1, x_2) = \min\{9 + x_1 + x_2, 10 + 2x_1 - x_2, 20 - 2x_1\}$, consider the optimization problem

$$(*) \qquad \text{maximize } h(x_1, x_2) \qquad \text{subject to} \quad x_1 \geq 0,\ x_2 \geq 0.$$

(a) Write a standard-form Linear Program equivalent to $(*)$. (Just write the LP: don't solve it yet!)

(b) Write the dual problem corresponding to the LP in part (a), and use Complementary Slackness to prove that the choices $x_1^* = 3$, $x_2^* = 2$ achieve the maximum in (a).

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(c) Write the optimal dictionary for the problem in (a). [The next page is blank in case you need it.]

(d) Solve the following modification of problem $(*)$:

$$(**) \qquad \text{maximize } h(x_1, x_2) \qquad \text{subject to} \quad x_2 \leq \tfrac{1}{2},\ x_1 \geq 0,\ x_2 \geq 0.$$

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(Blank page for extra calculations.)

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[15] **3.** Archaeologists digging in the Nile valley have found the inscription below chiselled into a large block of limestone that has been buried for centuries.

$$\left[\begin{array}{llr} \text{Maximize } f = & 3x_1 + 2x_2 + x_3 + 4x_4 & \\ \text{subject to} & x_1 + 3x_2 + x_3 + 3x_4 \le & 40 \\ & -2x_1 - 3x_2 + x_3 + 3x_4 \le & 8 \\ & x_2 \qquad - x_4 \le & -5 \\ & x_1, x_2, x_3, x_4 \ge & 0 \end{array}\right] \implies \left[\begin{array}{l} f = 95 - 3x_5 - 5x_7 - \\ \hline x_6 = \\ \\ \\ \\ \end{array}\right.$$

They can read the left part easily: it's a linear programming problem. But the block is broken, and only a fragment of the part on the right has survived. Assuming that the relic once showed a **correct optimal dictionary,** help the historians reconstruct the key missing ingredients.

You may assume that the ancients used x_5, x_6, x_7 as slack variables for the constraints in the order shown, but **you may not execute any simplex pivots**.

(a) Write the dual problem.

(b) Find the optimal value in the dual problem.

(c) Find an optimal input vector $\mathbf{y}^*$ for the dual problem.

(d) Find an optimal input vector $\mathbf{x}^*$ for the original problem.

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(e) Identify the variables that are basic in the incomplete optimal dictionary.

(f) Reconstruct the incomplete objective row in the optimal final dictionary.

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[15] **4.** **Use the Revised Simplex Method (RSM)** to find all solutions (if any) for the problem

$$\max\left\{f=\mathbf{c}^T\mathbf{x}\ :\ A\mathbf{x}=\mathbf{b},\ \mathbf{x}\geq\mathbf{0}\text{ in }\mathbb{R}^5\right\},$$

where

$$\mathbf{c}^T=\begin{bmatrix} 1 & -2 & 1 & 0 & 0\end{bmatrix},$$

$$A=\begin{bmatrix} 1 & 2 & -1 & -1 & 0\\ 1 & -4 & 1 & 0 & 1\end{bmatrix},\qquad \mathbf{b}=\begin{bmatrix}4\\2\end{bmatrix}.$$

Start with the basis $\mathcal{B}=\{2,5\}$, and follow the steps below.

(a) Find the current Basic Solution $\mathbf{x}$, and its objective value.

(b) Find the next entering variable (if there is one).

(c) Find the next leaving variable (if there is one).

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(d) Find the new basic feasible solution (BFS) after one pivot, and give its objective value.

(e) Is the problem now solved? If so, summarize your findings; if not, cycle back to step (b).
[Note: Complete at most one more RSM iteration. If this does not solve the problem, stop anyway.]

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[15] **5.** Cleo and Rory play a zero-sum game. Cleo, the column player, chooses a probability vector $\mathbf{x}$ in $\mathbb{P}(3)$; Rory, the row player, chooses a probability vector $\mathbf{y}$ in $\mathbb{P}(4)$. Then Rory pays $\mathbf{y}^T A\mathbf{x}$ to Cleo, where

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 4 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

(a) Explain why the actions of both players can be predicted by studying a certain 3×2 matrix A_0 instead of the given 4×3 matrix A.

(b) Set up a linear program to find Cleo's optimal probability vector $\mathbf{x}^*$. (You may use the simplified problem from part (a), or you may start from game as presented initially. Whatever you do, answer in terms of the given game setup, so that $\mathbf{x}^* \in \mathbb{P}(3)$.)

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(c) Solve the LP in part (b). You may use any method, but you must explain your approach.

(d) Find Rory's optimal probability vector $\mathbf{y}^*$. Check your answer; remember the instructions in part (b).

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[20] **6.** The hundredth anniversary of the Tour de France, the world's most famous cycling race, starts in just two days. Every racer is backed by a team of mechanics, trainers, and coaches who are optimizing every conceivable aspect of the next three weeks on the road. On long days, riders need to eat while they pedal. Our job is to plan a rider's snack pack—some mixture of nuts (x_1 grams), candy (x_2 grams), and "vitamins" (x_3 grams) that deliver the maximum energy boost. This energy, with units of kiloJoules (kJ), is

$$f = 4x_1 + 9x_2 + 5x_3.$$

Various limitations apply. The packet must not take up too much volume, so we insist on

$$x_1 + 2x_2 + x_3 \le 6, \qquad \text{each term in deciliters, or d}\ell. \tag{1}$$

Biochemical considerations for the post-race medical review require

$$2x_1 + 5x_2 + 3x_3 \le 15, \qquad \text{each term in milligrams, or mg.} \tag{2}$$

Finally, we don't want too many oxidants, so we require

$$3x_1 + 6x_2 + 4x_3 \le 19, \qquad \text{each term in International Units, or IU.} \tag{3}$$

The numbers above are not exact; however, they have a convenient relationship to the matrix identity

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 6 & 4 \end{bmatrix} \iff B^{-1} = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix}.$$

(a) Write a standard linear programming problem that expresses the optimization problem above.

(b) Write the dual problem for the problem in part (a).

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(c) Confirm that an optimal snack mix involves positive quantities of nuts, candy, and vitamins. Identify the amounts involved.

(d) Is the optimal mixture unique? Discuss.

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(e) A new company offers an edible synthetic product with an interesting profile: x_4 grams of this mystery substance would add $7x_4$ to the athlete's energy function, but occupy $2x_4$ dℓ of volume, add $2x_4$ mg of limited biochemicals, and contribute $3x_4$ IU of oxidants. Should we include some of this new product in an optimized snack pack?

(f) Predict the change in the rider's energy gain, f_{MAX}, if the allocation vector $\mathbf{b} = (6, 15, 19)$ in lines (1)–(3) is changed to $\mathbf{b}' = (6 + p_1, 15 + p_2, 19 + p_3)$ for some perturbation vector $\mathbf{p} = (p_1, p_2, p_3)$ with small entries.

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(g) Calculate the actual change in the rider's energy gain, f_{MAX}, if the allocation vector $\mathbf{b} = (6, 15, 19)$ in lines (1)–(3) is changed to $\mathbf{b}' = (5, 14, 20)$. Compare the true value with the approximation suggested by part (f).

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[5] **7.** Given a vector $\mathbf{c}$ in $\mathbb{R}^n$ and a matrix A of shape $m \times n$, prove that **one and only one** of the following two sets is nonempty:

$$S_1 \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \mathbb{R}^n \, : \, A\mathbf{x} \geq \mathbf{0} \text{ in } \mathbb{R}^m, \; \mathbf{c}^T\mathbf{x} < 0\right\},$$

$$S_2 \stackrel{\text{def}}{=} \left\{\mathbf{y} \in \mathbb{R}^m \, : \, \mathbf{y}^T A = \mathbf{c}^T, \; \mathbf{y} \geq \mathbf{0} \text{ in } \mathbb{R}^m\right\}.$$

The End