This examination has 14 pages including this cover.

UBC-SFU-UVic-UNBC Calculus Examination

6 June 2013, 12:00-15:00 PDT

Name: ______________________ **Signature:** ______________________

School: ______________________ **Candidate Number:** ______________________

Rules and Instructions

1. *Show all your work!* Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.
3. Any calculator compatible with the BC Ministry of Education's *2012/2013 Calculator Policy* may be used.
4. Some basic formulas appear on page 2. No other notes, books, or aids are allowed. In particular, *all calculator memories must be empty when the exam begins.*
5. If you need more space to solve a problem on page n, work on the back of page $n-1$.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

1		42
2		10
3		10
4		10
5		10
6		12
7		6
Total		100

UBC-SFU-UVic-UNBC Calculus Examination

Formula Sheet for 6 June 2013

Exact Values of Trigonometric Functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Trigonometric Definitions and Identities

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$\csc\theta = \dfrac{1}{\sin\theta}$

Continued on page 3

[42] **1. Short-Answer Questions.** Write your answers in the boxes provided. Each question is worth 3 marks, but not all questions have equal difficulty. *In this question, and throughout this exam, the answers have no point value unless correct supporting work is also shown.*

(a) Evaluate $\lim\limits_{x\to 9} \dfrac{x\,(\sqrt{x}-3)}{x-9}$.

ANSWER:

$$\frac{3}{2}$$

Detail: $\lim\limits_{x\to 9} \dfrac{x\,(\sqrt{x}-3)}{x-9} = \lim\limits_{x\to 9} \dfrac{x\,(\sqrt{x}-3)}{(\sqrt{x}-3)\,(\sqrt{x}+3)} = \lim\limits_{x\to 9} \dfrac{x}{(\sqrt{x}+3)} = \dfrac{9}{3+3}$.

(b) Find $\displaystyle\int \frac{dx}{e^{4x}}$.

ANSWER:

$$-\frac{1}{4}e^{-4x} + C,\ C \in \mathbb{R}$$

Detail: $\displaystyle\int \frac{dx}{e^{4x}} = \int e^{-4x}\,dx = \frac{e^{-4x}}{-4} + C.$

(c) Find y', given $y = (e^{3x}+x)^{2013}\cos(x)$.

ANSWER:

$$y' = 2013(e^{3x}+x)^{2012}(3e^{3x}+1)\cos(x) - (e^{3x}+x)^{2013}\sin(x)$$

Detail: This combines the product rule and the chain rule.

Continued on page 4

(d) Find the derivative of $f(x) = \dfrac{x^7}{\sin(x)}$.

ANSWER:

$$f'(x) = \frac{7x^6 \sin(x) - x^7 \cos(x)}{\sin^2(x)}$$

Detail: This is a direct application of the quotient rule.

(e) The function $f(x) = b/(x^2 + ax + 2)$ has a local maximum at $x = 1$, and the local maximum value $f(1)$ equals 2. Find the values of a and b.

ANSWER:

$$a = -2,\ b = 2$$

Detail: Here $f'(x) = -\dfrac{b(2x+a)}{(x^2+ax+2)^2}$. A local max at $x = 1$ requires $f'(1) = 0$, so $a = -2$. The prescribed value gives $2 = f(1) = b/(3+a)$.

(f) Let $a = \sqrt{100 - (8 \times 10^{-15})}$ and $b = 10 - (4 \times 10^{-16})$. Choose the correct statement and write it in the box: $a < b$, $a = b$, $a > b$. Explain your choice in the work-space provided.

ANSWER:

$$a < b$$

Detail: Note that $b^2 = (10)^2 - 2(10)(4 \times 10^{-16}) + (4 \times 10^{-16})^2 = a^2 + (4 \times 10^{-16})^2$. Thus $a^2 < b^2$, so $|a| > |b|$. Both numbers are positive.

Alternatively, let $f(x) = \sqrt{100 - x}$, so $f'(x) = -\dfrac{1}{2\sqrt{100-x}}$ and $f''(x) = -\dfrac{1}{4(100-x)^{3/2}}$. Since $f''(x) > 0$ whenever $x \in (0, 100)$, f is concave down in this interval, giving

$$f(x) < L(x) \stackrel{\text{def}}{=} f(0) + f'(0)(x - 0) = 10 - \frac{x}{20}, \qquad 0 < x < 100.$$

Plug in $x = 8 \times 10^{-15}$ to get $a < b$ from this.

Continued on page 5

(g) Given that $f'(x) = 2x - (3/x^4)$ for $x > 0$, and $f(1) = 3$, find $f(x)$ for $x > 0$.

ANSWER:

$$f(x) = x^2 + \frac{1}{x^3} + 1$$

Detail: The general antiderivative is $f(x) = x^2 + \dfrac{1}{x^3} + C$. Then $3 = f(1) = 2 + C$ forces $C = 1$.

(h) Find the equation of the line that is tangent to the curve $xy^2 + 2xy = 8$ at the point $(1, 2)$.

ANSWER:

$$y = 2 - \frac{4}{3}(x - 1)$$

Detail: Implicit differentiation gives $y^2 + 2xyy' + 2y + 2xy' = 0$. At $(x, y) = (1, 2)$, this says $4 + 4y' + 4 + 2y' = 0$, giving $y' = -4/3$. Use the point-slope form of the line.

(i) Find the (x, y) coordinates of the highest point on the curve $y = \dfrac{8x}{1 + 4x^2}$.

ANSWER:

$$\left(\frac{1}{2}, 2\right)$$

Detail: The quotient rule gives $y' = \dfrac{8(1 - 2x)(1 + 2x)}{(1 + 4x^2)^2}$. The critical points are at $x = \pm\frac{1}{2}$. Since the sign of y matches the sign of x, the high point must have $x > 0$. So use $x = \frac{1}{2}$; plug in to find $y = 2$.

Continued on page 6

(j) Find all intervals, if any, on which the function below is increasing:

$$f(x) = \frac{1 + \ln(x+1)}{x+1}, \quad x > -1.$$

ANSWER:

The single interval $(-1, 0]$

Detail: Here $f'(x) = -\dfrac{\ln(x+1)}{(x+1)^2}$ and we have

$$f'(x) > 0 \iff \ln(x+1) < 0 \iff x+1 < 1 \iff x < 0.$$

Since f is continuous at 0, we can include that point.

(k) Find the constant k that satisfies these simultaneous conditions for some function y:

$$\frac{y'}{y} = k, \quad y(0) = 2, \quad y(10) = 100.$$

ANSWER:

$$k = \frac{1}{10}\ln(50) = \frac{\log(50)}{10}$$

Detail: $y' = ky$ forces $y = Ae^{kx}$ for some constant A. Then $2 = y(0) = A$ leads to $y = 2e^{kx}$, and $100 = y(10) = 2e^{10k}$ requires $50 = e^{10k}$. Isolate k and box the answer.

(l) A certain function $f(x)$ with $f(1) = 4$ satisfies the identity

$$f'(x) = x\,(f(x))^2 + e^x, \qquad x > 0.$$

Find $f''(1)$.

ANSWER:

$$144 + 9e$$

Detail: Differentiate both sides to get $f''(x) = (f(x))^2 + x(2f(x)f'(x)) + e^x$. Substitute $x = 1$ to get $f''(1) = 16 + e + 8f'(1)$. But from the given equation, $f'(1) = f(1)^2 + e = 16 + e$. Back-substitute into $f''(1)$ and box the result.

Continued on page 7

(m) Given a function $f(x)$ that satisfies $f'(3) = 5$, evaluate this limit or determine that it does not exist:

$$\lim_{x\to 3} \frac{x^2 - 3x}{f(3) - f(x)}.$$

ANSWER:

$$-\frac{3}{5}$$

Detail: By definition, $5 = f'(3) = \lim_{x\to 3} \dfrac{f(x) - f(3)}{x - 3}$. Hence

$$-\frac{1}{5} = -\lim_{x\to 3} \frac{x-3}{f(x) - f(3)} = \lim_{x\to 3} \frac{x-3}{f(3) - f(x)}.$$

Use this together with the identity $\dfrac{x^2 - 3x}{f(3) - f(x)} = x\left(\dfrac{x-3}{f(3) - f(x)}\right)$.

(n) Find the constant k for which the limit L exists; write the values for both k and L in the answer box:

$$L = \lim_{x\to -\infty} \left(\sqrt{4x^2 + 3x} + kx\right).$$

ANSWER:

$$k = 2,\ L = -\frac{3}{4}$$

Detail: For the limit to exist as $x \to -\infty$, the term kx must be negative (because the other term is positive). Therefore we must have $k > 0$. Now

$$L = \lim_{x\to -\infty} \frac{\left(\sqrt{4x^2 + 3x} + kx\right)\left(\sqrt{4x^2 + 3x} - kx\right)}{\left(\sqrt{4x^2 + 3x} - kx\right)} = \lim_{x\to -\infty} \frac{4x^2 + 3x - k^2x^2}{\left(\sqrt{4x^2 + 3x} - kx\right)}.$$

Choose $k = 2$ to make the numerator and denominator orders match, then simplify

$$L = \lim_{x\to -\infty} \frac{3x}{\sqrt{4x^2 + 3x} - 2x} = \lim_{x\to -\infty} \frac{3x}{(-x)\sqrt{4 + 3/x} - 2x} = \lim_{x\to -\infty} \frac{3}{-\sqrt{4 + 3/x} - 2}.$$

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Full-Solution Questions. In questions 2–7, justify your answers and show all your work. Simplification is not required unless explicitly requested.

[10] **2.** A metal can in the shape of a cylinder with no top must be made to hold 64π cm^3 of liquid. Find the dimensions of the can that minimize the area of the metal required. Be sure to show that your design is a true minimizer.
Hint: The metal used consists of a circle (the bottom of the can) and a rectangle (the sides of the can).

Let r denote the radius of the can's base disk; let w and h denote the width and height of the metal rectangle that will form the can's sides. (Sketch!) The can's volume will then be

$$V = \pi r^2 h \qquad \left(\text{i.e.,} \quad h = \frac{V}{\pi r^2}\right).$$

The perimeter of the can's circular base, namely $2\pi r$, must match the width of the rectangle, w, so

$$w = 2\pi r.$$

The total area of metal used is

$$A = \pi r^2 + wh = \pi r^2 + (2\pi r)h = \pi r^2 + (2\pi r)\left(\frac{V}{\pi r^2}\right) = \pi r^2 + \frac{2V}{r}.$$

This defines a single-variable function $A = A(r)$ we seek to minimize over its natural domain, $r > 0$. We have

$$\frac{dA}{dr} = 2\pi r - \frac{2V}{r^2} = \frac{2\pi}{r^2}\left(r^3 - \frac{V}{\pi}\right).$$

The factor in front of the parentheses is positive, so we have

$$\frac{dA}{dr} < 0 \quad \text{for } 0 < r < \left(\frac{V}{\pi}\right)^{1/3},$$
$$\frac{dA}{dr} > 0 \quad \text{for } r > \left(\frac{V}{\pi}\right)^{1/3}.$$

It follows that A is minimized when $r = (V/\pi)^{1/3}$, which corresponds to

$$h = \frac{V/\pi}{r^2} = \frac{(V/\pi)}{(V/\pi)^{2/3}} = \left(\frac{V}{\pi}\right)^{1/3} = r.$$

For the special case $V = 64\pi$ cm^3, this gives

$$h = 4 \text{ cm} = r.$$

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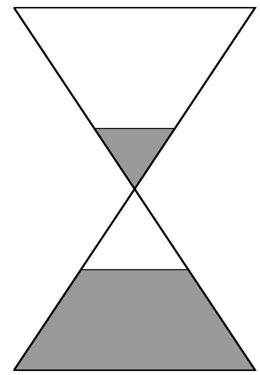
[10] **3.** An idealized hourglass is made by joining two identical circular cones, each with base radius 12 cm and height 18 cm, at their vertices. A tiny hole allows water to drop from the upper cone into the lower one. At a certain instant, the depth of water in the upper cone is 6 cm, this depth is decreasing at a rate of 0.2 cm/min, and the water in the lower cone is 10 cm deep. How fast is the depth of water increasing in the lower cone? (The sketch below shows a snapshot of the situation.)
Hint: The volume of a right circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.



Let r_1 and h_1 be the radius and height of the conical volume of water in the upper cone. (Please add these labels to the sketch provided.) Note that, by similarity with the container,

$$\frac{r_1}{h_1} = \frac{R}{H} = \frac{12}{18} = \frac{2}{3}, \qquad \text{so} \qquad r_1 = \frac{2}{3}h_1.$$

Thus the volume of water in the upper cone is

$$V_1 = \frac{1}{3}\pi r_1^2 h_1 = \frac{1}{3}\pi\left(\frac{2}{3}h_1\right)^2 h_1 = \frac{4}{27}\pi h_1^3.$$

At the instant of interest, the given information reveals

$$\frac{dV_1}{dt} = \frac{4}{9}\pi h_1^2 \frac{dh_1}{dt} = \frac{4}{9}\pi(6^2)(-0.2) = -\frac{16}{5}\pi.$$

That is, the volume of water in the upper chamber is decreasing at a rate of 3.2π cm^3 per minute.

The same ideas apply to the cone-shaped space above the water in the lower chamber. If we write r_2 and h_2 for the radius and height of this space, and V_2 for its volume, we have (just as before)

$$V_2 = \frac{4}{27}\pi h_2^3, \qquad \frac{dV_2}{dt} = \frac{4}{9}\pi h_2^2 \frac{dh_2}{dt}.$$

Focus on the instant of interest. Since there are 10 cm of water in the lower chamber, we have $h_2 = 18 - 10 = 8$. Water entering from the upper chamber is making V_2 decrease at a rate of 3.2π cm^3 per minute. Hence

$$-3.2\pi = \frac{4}{9}\pi(8^2)\frac{dh_2}{dt} \implies \frac{dh_2}{dt} = -\frac{9}{80}.$$

The height of the space in the lower chamber is shrinking at 9/80 cm per minute. This is the rate at which the depth of the water in that chamber is rising.

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[10] **4.** The displacement of a point moving along a straight line is given by $s = f(t)$, $t \geq 0$, where t is measured in seconds and s in meters. The point's acceleration function is $a(t) = 10 - 4t$. The point's initial position is $f(0) = 0$, and the point's velocity equals 0 at the instant when $t = 2$.

(a) Find the position function $s = f(t)$.

The point's velocity obeys $v'(t) = a(t) = 10 - 4t$, so $v(t) = 10t - 2t^2 + C$.

Using $0 = v(2) = 12 + C$ gives $C = -12$, so $v(t) = 10t - 2t^2 - 12$.

The position function obeys $f'(t) = v(t) = 10t - 2t^2 - 12$, so $f(t) = 5t - \dfrac{2}{3}t^3 - 12t + K$.

Using $0 = f(0) = K$ gives the answer: $s = f(t) = 5t - \dfrac{2}{3}t^3 - 12t$.

(b) Find the total distance travelled by the point during the first 3 seconds.

Factoring the velocity shows when the moving point changed direction:

$$v(t) = -2(t^2 - 5t + 6) = -2(t-2)(t-3).$$

The point moved to the left from $f(0) = 0$ to $f(2) = -\dfrac{28}{3}$ during the interval $[0, 2]$, then moved to the right from $f(2) = -\dfrac{28}{3}$ to $f(3) = -9$ during the interval $[2, 3]$. The total distance moved (in meters) was

$$[f(0) - f(2)] + [f(2) - f(3)] = \frac{28}{3} + \frac{1}{3} = \frac{29}{3}.$$

(c) Express the total distance travelled by the point as a function of t, valid for all $t \geq 0$.

Let $D(t)$ denote the total distance moved on the interval $[0, t]$. In terms of the function f already written,

$$D(t) = \begin{cases} -f(t), & \text{if } 0 \leq t \leq 2, \\ D(2) = \dfrac{28}{3}, & \text{if } t = 2, \\ D(2) + [f(t) - f(2)], & \text{if } 2 \leq t \leq 3, \\ D(3) = \dfrac{29}{3}, & \text{if } t = 3, \\ D(3) + [f(3) - f(t)], & \text{if } 3 \leq t. \end{cases}$$

Recall $f(2) = -\dfrac{28}{3}$ and $f(3) = -9$ to make this completely explicit.

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[10] **5.** Consider the plane curve $y = x^3 + 1$. Give this curve the name $\mathcal{C}$.

(a) The line that is tangent to $\mathcal{C}$ at the point where $x = 3$ passes through $(a, 0)$. Find a.

Using $y' = 3x^2$ and $x = 3$, the line of interest has slope $m = 27$ and equation

$$y = 28 + 27(x - 3).$$

The point $(a, 0)$ lies on this line when

$$0 = 28 + 27(a - 3), \quad \text{i.e.,} \quad a = 3 - \frac{28}{27} = 2 - \frac{1}{27} = \frac{53}{27} = 1.962\,962\,962\,\ldots.$$

(b) We are looking for a point on $\mathcal{C}$ from which the tangent line passes through the point $(2, 0)$. Show how to use Newton's method to find the x-coordinate of such a point, and calculate two Newton steps starting from the initial guess $x = 3$.

(If you have a calculator, report decimal answers. If you don't, present a simplified rational answer for the first step and a "calculator-ready" expression for the second.)

Repeat the steps above, but let the point of tangency be $x = p$ instead of $x = 3$. Then the tangent line is

$$y = (p^3 + 1) + (3p^2)(x - p) = 1 + 3p^2 x - 2p^3.$$

This line passes through $(x, y) = (2, 0)$ if and only if $0 = 1 + 6p^2 - 2p^3$. We are to determine p from this equation.

To make this more familiar, let $x = p$ and define

$$f(x) = 1 + 6x^2 - 2x^3; \qquad \text{note that } f'(x) = 12x - 6x^2.$$

Newton's method takes a starting guess x_0 and improves it to

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{6x_0^2 - 2x_0^3 + 1}{12x_0 - 6x_0^2}.$$

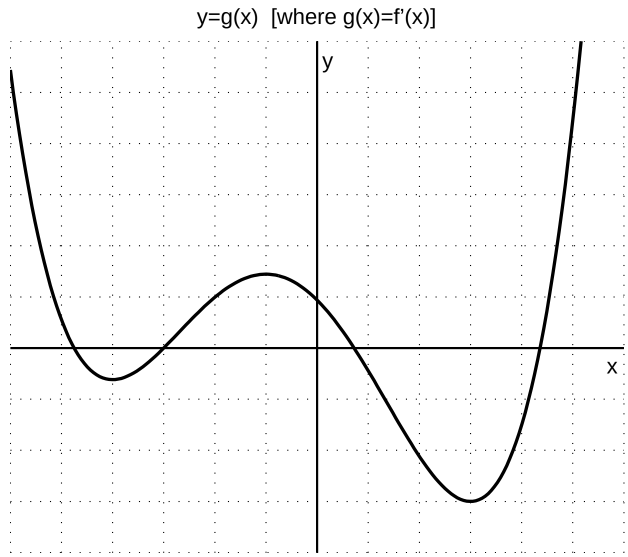
In particular,

$$x_0 = 3 \implies x_1 = 3 - \frac{54 - 54 + 1}{36 - 54} = 3 + \frac{1}{18} = \frac{55}{18} \approx 3.055\,555;$$

$$x_0 = \frac{55}{18} \implies x_1 = \frac{55}{18} - \frac{6x_0^2 - 2x_0^3 + 1}{12x_0 - 6x_0^2} \approx 3.053\,624.$$

Continued on page 12

[12] **6.** A certain function f satisfies $f(-1) = 0$ and $f'(x) = g(x)$, where the graph of g is shown below. Assume that g is positive-valued and concave up at all points not shown in the sketch. (The spacing between gridlines in the sketch is 1 unit.)



Answer questions (a)–(c) below without attempting to find an algebraic formula for $f(x)$.

(a) Determine the intervals where f is increasing or decreasing, and the (x, y)-coordinates of all local maxima and minima (if any).

The function f is increasing on intervals where $g(x) = f'(x) > 0$, namely,

$$(-\infty, -4.7), \quad (-3, 0.6), \quad \text{and} \quad (4.4, +\infty).$$

The function f is decreasing on intervals where $g(x) = f'(x) < 0$, namely,

$$(-4.7, -3) \quad \text{and} \quad (0.6, 4.4).$$

Thus f has local maxima at the points where $x \approx -4.7$ and $x \approx 0.6$, and f has local minima at points where $x = -3$ and $x \approx 4.4$.

The Fundamental Theorem of Calculus says that

$$f(x) - f(-1) = \int_{-1}^{x} f'(t)\,dt, \quad \text{i.e.,} \quad f(x) = \int_{-1}^{x} g(t)\,dt.$$

Rough estimates of area under the graph shown above lead to

$$f(0.6) \approx 1.6, \qquad f(4.4) \approx 1.6 - 6 = -4.4,$$
$$f(-3) \approx -1.8, \qquad f(-4.8) \approx -1.8 + 0.6 = -1.2.$$

Continued on page 13

(b) Determine the intervals where f is concave up or down, and the (x, y)-coordinates of all inflection points (if any).

The function f is concave up on intervals where $g(x) = f'(x)$ is increasing, namely,

$$(-4, -1) \quad \text{and} \quad (3, +\infty).$$

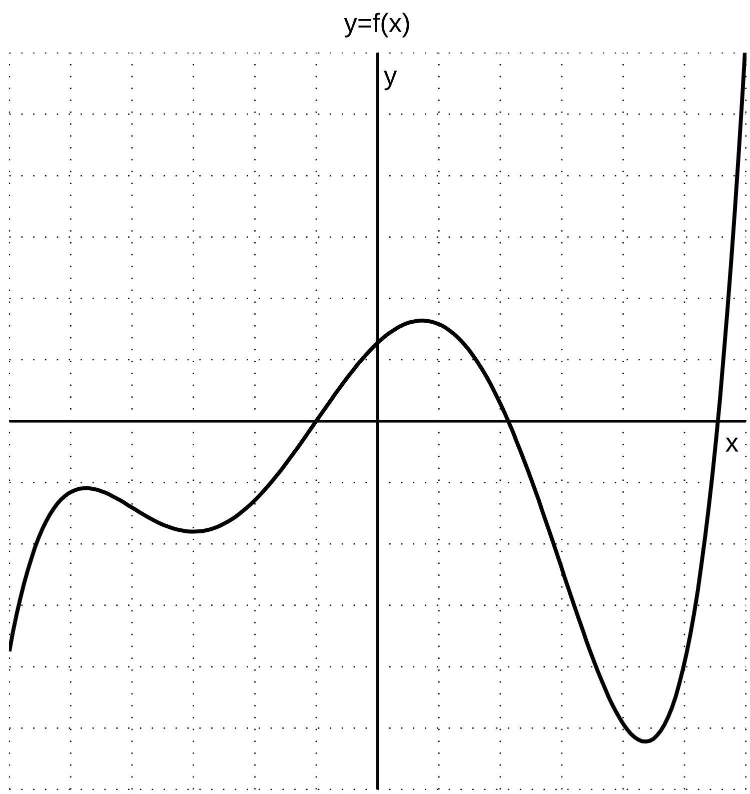
The function f is concave down on intervals where $g(x) = f'(x)$ is decreasing, namely,

$$(-\infty, -4) \quad \text{and} \quad (-1, 3).$$

Thus f has inflections at the points where $x = -4$, $x = -1$, and $x = 3$. Considering areas under the given curve leads to the following approximate coordinates for the inflection points:

$$(-4, -1.5), \quad (1, 0), \quad (3, -1.6).$$

(c) Sketch the curve $y = f(x)$, showing the features given in items (a)–(b) above and giving the (x, y) coordinates of all points occurring above and also all x-intercepts (if any).



The information in (a)–(b) is enough to produce a very good approximation to the graph above. The exact function f used in this question, not requested, was

$$f(x) = \frac{1}{144}\left[\frac{3}{5}x^5 + 2x^4 - 22x^3 - 72x^2 + 135x + \frac{918}{5}\right].$$

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[6] **7.** Prove both statements below. Include the names of any famous theorems you rely on.

(a) The equation $x + \ln|x| = 0$ has at least one solution for x in the open interval $(-1, 1)$.

Let $f(x) = x + \ln|x|$. This is continuous on the interval $[1/e, 1]$, with $f(1/e) = e^{-1} - 1 < 0$ and $f(1) = 1 > 0$. By the Intermediate Value Theorem, there must be at least one point x in the open interval $(1/e, 1)$ where $f(x) = 0$. Of course, any such point must also lie in the specified interval, $(-1, 1)$.

(b) The equation $x + \ln|x| = 0$ has exactly one solution for x in the open interval $(-1, 1)$.

With $f(x) = x + \ln|x|$ as above, note that $f'(x) = 1 + 1/x$.

- Since $f'(x) < 0$ whenever $x \in (-1, 0)$, f is decreasing on the interval $[-1, 0)$. We have $f(-1) = -1$, so $f(x) < -1$ for each $x \in (-1, 0)$. This interval will contribute no solutions.
- The function f is undefined when $x = 0$. So $x = 0$ cannot be a solution.
- We have $f'(x) > 0$ whenever $x \in (0, 1)$, so f is increasing on the interval $(0, 1]$. (This is a consequence of the Mean Value Theorem.) In particular, there can be at most one x in this interval where $f(x) = 0$. (One could also cite Rolle's Theorem for this.)

The End