Solutions.

SFU-UBC-UNBC-UVic

CALCULUS CHALLENGE EXAMINATION

JUNE 07, 2012, 12:00 - 15:00

Name: ______________________________ (please print)
family name *given name*

Signature: ______________________________

Instructions:

1. Do not open this booklet until told to do so.
2. Write your name above in block letters and sign your exam.
3. Write your answer in the space provided below the question. If additional space is needed then use the back of the previous page. Your final answer should be simplified as far as is reasonable.
4. Make the method you are using clear in every case unless it is explicitly stated that no explanation is needed.
5. This exam has 15 questions on 15 pages (not including this cover page and the sheet of trigonometric identities). Once the exam begins please check to make sure your exam is complete.
6. **No** Books, papers, or electronic devices except a graphing calculator and the sheet of trigonometric identities attached to this examination shall be within the reach of a student during the examination.
7. **During the examination, communicating with, or deliberately exposing written papers to the view of, other examinees is forbidden.**

Question	Maximum	Score
1	8	
2	11	
3	8	
4	6	
5	8	
6	11	
7	6	
8	5	
9	6	
10	8	
11	6	
12	8	
13	8	
14	8	
15	8	
Total	115	

1. Compute the following limits.

[2] (a) $\lim_{x\to\infty} \frac{\ln x}{e^{2x}}$ $\left[\frac{\infty}{\infty}\right]$

$\overset{l'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2e^{2x}} = \lim_{x\to\infty} \frac{1}{2x\,e^{2x}}$ $\boxed{= 0}$

[3] (b) $\lim_{x\to 2}(x-1)^{\frac{1}{x-2}}$ $[1^\infty]$ $L = \lim_{x\to 2} (x-1)^{1/x-2}$

$\ln L = \ln \lim_{x\to 2} (x-1)^{\frac{1}{x-2}} = \lim_{x\to 2} \ln (x-1)^{\frac{1}{x-2}}$

$\ln L = \lim_{x\to 2} \frac{\ln(x-1)}{x-2}$ $\left[\frac{0}{0}\right]$ $\overset{l'H}{=} \lim_{x\to 2} \frac{\frac{1}{x-1}}{1} = 1$

$\ln L = 1$ $\therefore$ $L = e$

and $\lim_{x\to 2} (x-1)^{\frac{1}{x-2}}$ $\boxed{= e.}$

[3] (c) $\lim_{x\to 6} \frac{\sqrt{x-2}-2}{\sqrt{x-6}}$ $\left[\frac{0}{0}\right]$

$= \lim_{x\to 6} \frac{(\sqrt{x-2}-2)}{\sqrt{x-6}} \frac{(\sqrt{x-2}+2)}{(\sqrt{x-2}+2)} = \lim_{x\to 6} \frac{x-2-4}{(\sqrt{x-6})(\sqrt{x-2}+2}$

$= \lim_{x\to 6} \frac{x-6}{(\sqrt{x-6})(\sqrt{x-2}+2} = \lim_{x\to 6} \frac{(\sqrt{x-6})(\sqrt{x-6})}{(\sqrt{x-6})(\sqrt{x-2}+2)}$

$\boxed{= 0}$

2. Compute the indicated derivatives. You do not need to simplify your answer.

[3] (a) If $y = \frac{\cos x^2}{\sin x}$, find y'.

$$y' = \frac{(\sin x)(-\sin x^2)(2x) - \cos x^2 \cdot \cos x}{\sin^2 x}$$

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[4] (b) If $y = (1-3x)^{e^x}$, find $\frac{dy}{dx}$.

$$\ln y = \ln (1-3x)^{e^x} = e^x \ln(1-3x)$$

$$\frac{1}{y} y' = e^x \ln(1-3x) + e^x \cdot \frac{1}{1-3x}(-3)$$

$$y' = (1-3x)^{e^x} e^x \left[\ln(1-3x) - \frac{3}{1-3x}\right]$$

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[4] (c) Suppose that the functions f and g and their derivatives with respect to x have the values at $x = 0$ and $x = 1$ as shown in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	5	2	1	3
1	7	−3	−2	−5

Evauluate $\frac{d(f(x+g(x)))}{dx}$ at $x = 0$. $= f'(x+g(x))[1+g'(x)]$

At $x=0$ $f'(0+g(0))[1+g'(0)]$

$= f'(0+1)[1+3]$

$= f'(1) \cdot 4$

$= (-3)(4) \boxed{= -12}$

[2] **3.** (a) State the definition of the derivative of a function f AT A NUMBER a.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{if this limit exists}$$

OR

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{if this limit exists}$$

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[4] (b) Let $f(x) = \dfrac{3x+1}{x-2}$. Use the DEFINITION of the derivative to find $f'(1)$.

$$f(1) = \frac{(3)(1)+1}{1-2} = -4$$

$$f'(1) = \lim_{x \to 1} \frac{\frac{3x+1}{x-2} - (-4)}{x-1} = \lim_{x \to 1} \frac{3x+1+4(x-2)}{(x-2)(x-1)}$$

$$= \lim_{x \to 1} \frac{7x-7}{(x-1)(x-2)} = \lim_{x \to 1} \frac{7\cancel{(x-1)}}{\cancel{(x-1)}(x-2)} = -7$$

$$\boxed{f'(1) = -7}$$

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[2] (c) Find the equation of the tangent line to $y = f(x)$ at $x = 1$.

When $x = 1$, $f(1) = -4$, The point is $(1, -4)$

Slope is $f'(1) = -7$

Equation is $y - (-4) = -7(x-1)$

$$y + 4 = -7x + 7$$

$$\boxed{y = -7x + 3}$$

[2] **4.** (a) Find f if $f' = 2\cos x + 8x^3 - e^x$ and $f(0) = 7$

Taking the anti derivative.

$$f(x) = 2\sin x + 2x^4 - e^x + C$$

$$f(0) = 7 = 2\sin(0) + 2(0)^4 - e^0 + C$$

$$7 = -1 + C \quad \therefore \quad C = 8$$

$$f(x) = 2\sin x + 2x^4 - e^x + 8$$

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[4] (b) Find a curve $y = f(x)$ with the following properties:

1) $\dfrac{d^2y}{dx^2} = 6x$ and

2) Its graph passes through the point $(0,1)$ and has a horizontal tangent line there.

$$y'' = 6x$$

$$y' = 3x^2 + C$$

$$y = x^3 + Cx + D$$

$(0,1)$ is on the curve

$$1 = 0^3 + (C)(0) + D \quad \therefore \quad D = 1$$

Horizontal tangent line at $x=0$

$$y'(0) = 0 \Rightarrow 0 = 3(0)^2 + C \Rightarrow C = 0$$

$$\therefore \quad y = x^3 + 1$$

[6] **5.** (a) Use a right-endpoint approximation with $n = 4$ steps to estimate the integral

$a=0$
$b=2$
$n=4$

$$A = \int_0^2 (x^2 - x)dx.$$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$x_0 = 0$

right end points

$x_1 = 1/2 \quad f(1/2) = (1/2)^2 - 1/2 = 1/4 - 1/2 = -1/4$

$x_2 = 1 \quad f(1) = 1^2 - 1 = 0$

$x_3 = 3/2 \quad f(3/2) = (3/2)^2 - 3/2 = 9/4 - 3/2 = 9/4 - 6/4 = 3/4$

$x_4 = 2 \quad f(2) = 2^2 - 2 = 2$

$$A \approx \sum_{i=1}^{4} f(x_i)\Delta x = \frac{1}{2}\sum_{i=1}^{4} f(x_i) = \frac{1}{2}\left[-\frac{1}{4} + 0 + \frac{3}{4} + 2\right]$$

$$A \approx \left(\frac{1}{2}\right)\left(2 + \frac{1}{2}\right) = 5/4$$

The estimate is $5/4$.

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[2] (b) Find the exact value of the integral in part (a).

$$\int_0^2 (x^2 - x)dx = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_0^2$$

$$= \left(\frac{1}{3}\right)(8) - \left(\frac{1}{2}\right)(4) = \frac{8}{3} - 2 = \frac{8}{3} - \frac{6}{3} = \frac{2}{3}$$

$$\int_0^2 (x^2 - x)dx = \frac{2}{3}$$

[3] **6.** (a) State the **Intermediate Value Theorem** clearly identifying all the hypothesis and the conclusion.

Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$

[4] (b) Use the Intermediate Value Theorem to show that

$$e^x = \sqrt{x} + 2$$

for some $x > 0$.

for $x \geq 0$

Consider $f(x) = e^x - \sqrt{x} - 2$. $f(x)$ is continuous for $x \geq 0$ since it is a sum of continuous functions. We need to show that $f(x) = 0$ for some $x > 0$. Note $f(0) = e^0 - \sqrt{0} - 2 = 1 - 2 = -1 < 0$ and $f(4) = e^4 - \sqrt{4} - 2 = e^4 - 4 \approx 50.6 > 0$. Since $f(0) < 0$ and $f(4) > 0$ and $f(x)$ continuous, there exists some $x \in (0,4)$ such that $f(x) = 0$.

[4] (c) Find the value of b so that the function

$$f(x) = \begin{cases} x^3 + bx + 3 & \text{if } x \leq 2 \\ be^{x-2} & \text{if } x > 2 \end{cases}$$

is continuous everywhere. Justify your answer.

$f(x)$ is continuous on $(-\infty, 2)$ since polynomials are continuous on $\mathbb{R}$

$f(x)$ is continuous on $(2, \infty)$ since e^x is continuous on $\mathbb{R}$.

At $x = 2$, $f(2) = 2^3 + 2b + 3 = 11 + 2b$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^3 + bx + 3 = 11 + 2b$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} be^{x-2} = b$$

For continuity at $x = 2$, we must have $b = 11 + 2b$ or

$\underline{b = -11}$

7. Newton's method can be used to calculate $\sqrt[4]{15}$ by calculating the zeros of $f(x) = x^4 - 15$.

[4] (a) Find the iteration formula to calculate x_{n+1} from x_n. Simply this formula as much as possible.

$$f(x) = x^4 - 15 \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x) = 4x^3$$

$$x_{n+1} = x_n - \left(\frac{x_n^4 - 15}{4x_n^3}\right) = x_n - \frac{1}{4}x_n + \frac{15}{4x_n^3}$$

$$x_{n+1} = \frac{3x_n}{4} + \frac{15}{4x_n^3}$$

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[2] (b) Perform one iteration of Newton's method to calculate $\sqrt[4]{15}$ using an initial guess of $x_0 = 2$.

$$x_0 = 2$$

$$x_1 = \frac{3x_0}{4} + \frac{15}{4x_0^3}$$

$$x_1 = \frac{(3)(2)}{4} + \frac{15}{(4)(2)^3} = \frac{6}{4} + \frac{15}{32}$$

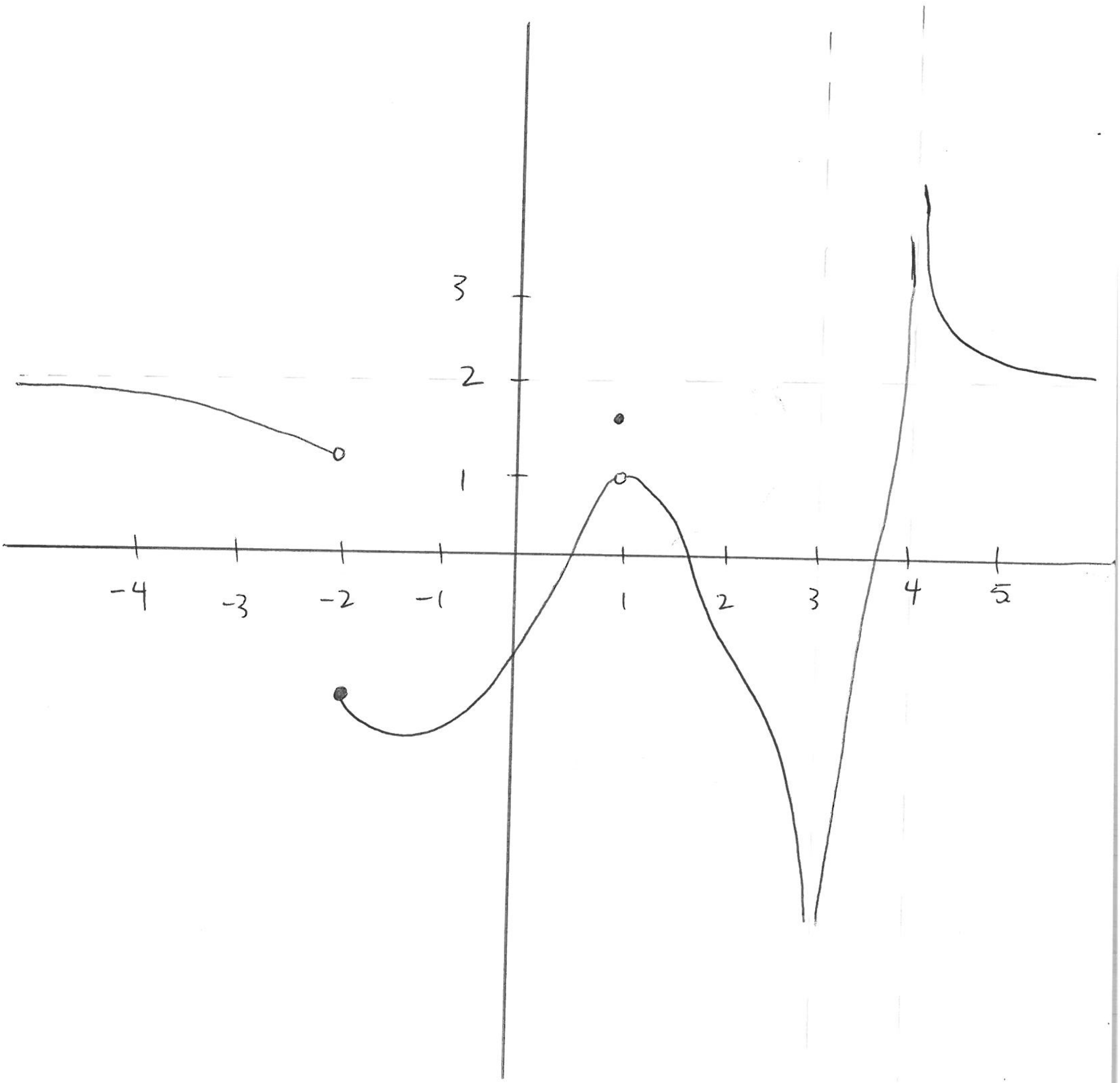
$$x_1 = \frac{48}{32} + \frac{15}{32}$$

$$\sqrt[4]{15} \approx x_1 = \frac{63}{32} \underbrace{\approx 1.96875}_{\text{no need for decimal approximation.}}$$

[5] **8.** Sketch the graph of a function $f(x)$ which meets all of the following criteria:

This is one of many possible answers.

1. $f(x)$ has exactly one horizontal asymptote with equation $y = 2$.
2. $f(x)$ has exactly two vertical asymptotes with equations $x = 3$ and $x = 4$.
3. $f(x)$ has a jump discontinuity at $x = -2$ and a removable discontinuity at $x = 1$.



[3] **9.** (a) Assume that f is a function such that $f(5) = 2$ and $f'(5) = 4$. Using a linear approximation to f near $x = 5$, find an approximation to $f(4.9)$

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

$$a = 5$$
$$x = 4.9$$

$$f(4.9) \approx f(5) + f'(5)(4.9-5)$$

$$f(4.9) \approx 2 + (4)(-0.1) = 2 - 0.4 = 1.6$$

$$\boxed{f(4.9) \approx 1.6}$$

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[3] (b) The volume of a spherical balloon is given by $V = \frac{4}{3}\pi r^3$. If you can measure the radius r to within an accuracy of 5%, how accurate is your calculation of the volume?

$V = 4/3 \pi r^3$ We are given $\frac{dr}{r} = 5\% = 0.05$

Question is to find $\frac{\Delta V}{V}$ which is approximately $\frac{dV}{V}$

$$\frac{dV}{dr} = \frac{4}{3} \cdot 3\pi r^2 = 4\pi r^2$$

$$dV = 4\pi r^2 dr$$

$$\frac{dV}{V} = \frac{4\pi r^2 dr}{V} = \frac{\cancel{4}\cancel{\pi}\cancel{r^2} dr}{\frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}}} = 3\frac{dr}{r}$$

$$\frac{dV}{V} = (3)(5\%) = 15\% \text{ or } 0.15$$

Your volume calculation has accuracy of $\pm 15\%$.

10. A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec reaches a height of $s = 24t - 0.8t^2$ metres in t sec.

$S = 24t - 0.8t^2$

[2] (a) Find the rock's velocity and acceleration at time t.

$v(t) = \frac{ds}{dt} = \frac{d}{dt}(24t - 0.8t^2) = 24 - 1.6t$ $\boxed{v(t) = 24 - 1.6t}$ m/s

$a(t) = \frac{dv}{dt} = \frac{d}{dt}(24 - 1.6t) = -1.6$ $a(t) = -1.6$ m/s^2

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[2] (b) How long does it take the rock to reach its highest point?

At the highest point $v(t) = 0$ or $24 - 1.6t = 0$

$t = 24/1.6 = 15$

$t = 15$ seconds

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[2] (c) How high does the rock go?

$S(15) = (24)(15) - (0.8)(15^2) = 180$ metres.

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[2] (d) How long is the rock aloft?

Find the times between when $s(t) = 0$

$s(t) = 0 = 24t - 0.8t^2 = t(24 - 0.8t)$

$t(24 - 0.8t) = 0$ when $t = 0$ (when it is thrown upwards)

and when $24 - 0.8t = 0$ or $t = 24/0.8 = 30$ seconds.

$\therefore$ The rock is aloft for 30 seconds.

[6] **11.** Find the equation of the tangent line to the curve $x^3 + y^3 - 9xy = 0$ at the point $(2, 4)$.

For the eq'n of the tangent line, we need the slope of the tangent line to the curve $x^3+y^3-9xy=0$ at the point $(2,4)$

Find $\frac{dy}{dx}$ when $x=2$, $y=4$

$x^3+y^3-9xy=0$ Differentiate wrt x.

$$3x^2+3y^2y'-9(y+xy')=0$$

$$3x^2+3y^2y'-9y-9xy'=0$$

$$3y^2y'-9xy'=-3x^2+9y$$

$$y'(3y^2-9x)=-3x^2+9y$$

$$y'=\frac{-3x^2+9y}{3y^2-9x}$$

At $(2,4)$ $$y'=\frac{(-3)(2)^2+(9)(4)}{(3)(4)^2-(9)(2)}=\frac{-12+36}{48-18}=\frac{24}{30}=\frac{4}{5}$$

Eq'n of tangent line

$$(y-4)=\frac{4}{5}(x-2).$$

[8] **12.** A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 min, the egg's temperature is 38°C. Assuming the the water has not warmed appreciably, use Newton's Law of Cooling to determine how much longer it will take the egg to reach 20°C?

Newton's Law of Cooling. $\frac{dT}{dt} = k(T - T_s)$

$T_s = 18°C$ $T(0) = 98°C$ $T(5\text{ min}) = 38°C$

$\frac{dT}{dt} = k(T-18)$ Let $y = T - 18$, then $\frac{dy}{dt} = \frac{dT}{dt}$

$\frac{dy}{dt} = ky$ which has solution $y(t) = y(0)e^{kt}$ $y(0) = T(0) - 18$
$y(0) = 98 - 18 = 80$

Putting in terms of T, we have

$T(t) - 18 = 80e^{kt}$

$T(t) = 80e^{kt} + 18$

$T(5) = 38 = 80e^{5k} + 18$

$20 = 80e^{5k}$

$4^{-1} = e^{5k}$ $-\ln 4 = 5k$ $k = -\frac{1}{5}\ln 4 = \ln 4^{-1/5}$

$T(t) = 80e^{(\ln 4^{-1/5})t} + 18$

When the temperature is 20°C, the time is t_{20}

$T(t_{20}) = 20 = 80e^{\ln 4^{-1/5} t_{20}} + 18$

$2 = 80e^{\ln 4^{-1/5} t_{20}}$

$40^{-1} = e^{\ln 4^{-1/5} t_{20}}$

$(-\ln 40) = (\ln 4^{-1/5})t_{20}$

$t_{20} = \frac{-\ln 40}{\ln 4^{-1/5}} \approx 13.3048$ min.

So, it takes an additional 13.3048 − 5 minutes or a further ≈ 8.3 min to reach 20°C.

[8] **13.** Consider a cube of variable size (the edge length is increasing). Assume that the volume of the cube in increasing at at the rate of 10 cm^3/minute. How fast is the surface area increasing when the edge length is 8 cm?

Let x be the edge length
Let V be the volume and
Let S be the surface area

$V(x) = x^3$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$10\,\text{cm}^3/\text{min} = (3)(8\text{ cm})^2 \, dx/dt$$

$$\frac{dx}{dt} = \frac{10}{(3)(64)} \text{ cm}/\text{min}.$$

$S(x) = 6x^2$

$$\frac{dS}{dt} = 12x \frac{dx}{dt}$$

$$\frac{dS}{dt} = \frac{(12)(8)(10)}{(3)(64)} = 5\text{ cm}^3/\text{min}.$$

The surface area is increasing at a rate of $5\text{ cm}^3/\text{min}$.

[8] **14.** There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per tree drops by 10 apples. How many trees should be added to the existing orchard in order to maximize the total number of apples produced in the orchard?

Let x equal the number of trees to be added

Then $x \in [0, 80]$

Let $A(x)$ be the number of apples produced

$$A(x) = (50 + x)(800 - 10x)$$

$$A(x) = 40{,}000 - 500x + 800x - 10x^2$$

$$A(x) = 40{,}000 + 300x - 10x^2$$

$\frac{dA}{dx} = 300 - 20x$ Critical numbers: $\frac{dA}{dx} = 0$ or $\frac{dA}{dx}$ does not exist.

$$\frac{dA}{dx} = 0 = 300 - 20x \Rightarrow x = 15$$

$\frac{dA}{dx}$ always exists

$A(15) = (65)(650) = 42{,}250$ apples

Check Endpoints $A(0) = (50)(800) = 40{,}000$ apples

$A(80) = (130)(0) = 0$ apples.

$\therefore$ You should add 15 apples trees.

[8] **15.** Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

$f'(x) = x^3$ means $f(x) = \frac{x^4}{4} + c$

$x + y = 0$ is tangent to $f(x)$ means that at some point (x_0, y_0), the graph of the tangent line is $y = -x$. The slope of the tangent line is therefore -1

$f'(x_0) = -1 = x_0^3 \quad \therefore \quad x_0 = -1$

Point-slope form of tangent line

$y - y_0 = m(x - x_0)$

$y - y_0 = (-1)(x - (-1)) = -x - 1$

Since $x + y = 0$ and $y + x = y_0 - 1$

we have $y_0 = 1$

The tangent line $x + y = 0$ is tangent to $f(x)$ at $(-1, 1)$

$f(x) = \frac{x^4}{4} + c$

$1 = \frac{(-1)^4}{4} + C \quad \therefore \quad C = 3/4$

$\therefore$ A function is $f(x) = \frac{x^4}{4} + \frac{3}{4}$

Formula Sheet

Exact Values of Trigonometric Functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Trigonometric Definitions and Identities

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$