[10] **1.** (a) Find $f'(x)$, but do not simplify, given $f(x) = \dfrac{\sin(x)\cos(x)}{1+e^{3x}}$.

$$f'(x) = \frac{(\cos^2 x - \sin^2 x)[1+e^{3x}] - \sin(x)\cos(x)\left[3e^{3x}\right]}{(1+e^{3x})^2}$$

(b) Find $\dfrac{dy}{dx}$, but do not simplify, given $y = \sin\left(\sqrt{1+x^4} - x^2\right)$.

$$\frac{dy}{dx} = \cos\left(\sqrt{1+x^4} - x^2\right)\left[\frac{4x^3}{2\sqrt{1+x^4}} - 2x\right]$$

(c) Find $\displaystyle\int \frac{dx}{1+25x}$.

$$\int \frac{dx}{1+25x} = \frac{1}{25}\ln(1+25x) + C.$$

(d) Find $\displaystyle\int \frac{dx}{1+25x^2}$.

$$\int \frac{dx}{1+25x^2} = \int \frac{dx}{1+(5x)^2} = \alpha \tan^{-1}(5x) + C$$

for some α. $\dfrac{d}{dx}\left(\alpha \tan^{-1} 5x\right) = \dfrac{5\alpha}{1+(5x)^2}$,

so $\alpha = \frac{1}{5}$.

ANS: $\frac{1}{5}\tan^{-1}(5x) + C$.

Continued on page 4

[9] **2.** Find the exact values of these limits. Show your work.

(a) $\lim\limits_{x \to 2} \dfrac{x^3-8}{x^2-4}$.

When $x \neq 2$, $\dfrac{x^3-8}{x^2-4} = \dfrac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)}$.

So $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2-4} = \dfrac{x^2+2x+4}{x+2}\Bigg|_{x=2} = \dfrac{12}{4} = \underline{\underline{3}}.$

(b) $\lim\limits_{x \to 0} \left(\dfrac{x}{\sqrt{1+7x}-1} \cdot \dfrac{\sqrt{1+7x}+1}{\sqrt{1+7x}+1} \right) = \lim\limits_{x\to 0} \dfrac{x\left(\sqrt{1+7x}+1\right)}{(1+7x)-1}$

$= \lim\limits_{x\to 0} \dfrac{\sqrt{1+7x}+1}{7} = \underline{\underline{\dfrac{2}{7}}}.$

(c) $\lim\limits_{x \to \infty} \dfrac{a^x}{1+a^x}$, where $a > 0$ is constant. (Explain how the answer depends on a.)

Case $0 < a < 1$: $a^x \to 0$ as $x \to \infty$, so

$\lim\limits_{x\to\infty} \dfrac{a^x}{1+a^x} = \dfrac{0}{1+0} = 0.$

Case $a = 1$: $a^x = 1$ whenever $x > 0$, so

$\lim\limits_{x\to\infty} \dfrac{a^x}{1+a^x} = \lim\limits_{x\to\infty} \dfrac{1}{1+1} = \dfrac{1}{2}.$

Case $a > 1$: $a^x \to \infty$ as $x \to \infty$, so

$\lim\limits_{x\to\infty} \dfrac{a^x}{1+a^x} \cdot \dfrac{1/a^x}{1/a^x} = \lim\limits_{x\to\infty} \dfrac{1}{\frac{1}{a^x}+1} = \dfrac{1}{0+1} = 1.$

Continued on page 5

[10] **3.** Both parts of this question concern the point P with coordinates $(1,2)$ on the curve

$$x^3 + y^3 = 3xy + 3. \qquad (*)$$

(a) Sketch and find the area of the triangle described as follows. One vertex is at the origin. The other two vertices are the x and y intercepts of the line that is tangent to the curve in $(*)$ at the point P.

(b) Calculate y'' at the point P for the curve in $(*)$. Use your answer to make a rough sketch that shows the relationship of the curve to its tangent line near P. (If you can combine this sketch with the one requested in part (a), please do so.)

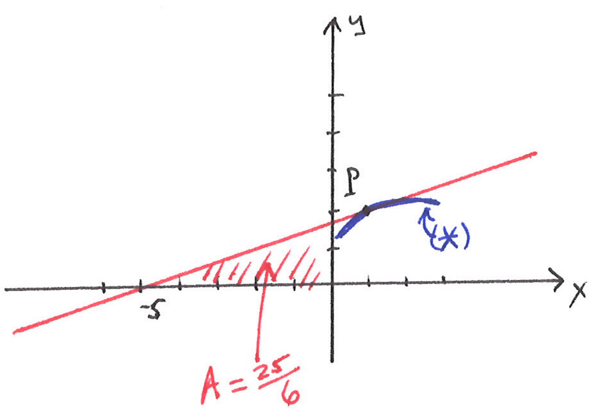
(a) On curve $(*)$, implicit differentiation gives

$$3x^2 + 3y^2 y' = 3y + 3xy' + 0, \text{ i.e.,}$$

$$x^2 + y^2 y' = y + xy'. \qquad (**)$$

So at P, $1 + 4y' = 2 + y' \Rightarrow 3y' = 1 \Rightarrow y' = 1/3$.

Tan line $y = 2 + \frac{1}{3}(x-1) = \frac{5}{3} + \frac{1}{3}x$ has intercepts $(0, 5/3)$ and $(-5, 0)$. Triangle area $A = \frac{1}{2}(5)\left(\frac{5}{3}\right) = \frac{25}{6}$.



(b) Differentiate again in $(**)$:

$$2x + (2y\,y')y' + y^2 y'' = y' + y' + xy''.$$

Plug in all we know at P, namely, $x=1$, $y=2$, $y' = \frac{1}{3}$:

$$2 + 4\left(\tfrac{1}{3}\right)^2 + 4y'' = \tfrac{1}{3} + \tfrac{1}{3} + y''.$$

Rearrange: $3y'' = \frac{2}{3} - 2 - \frac{4}{9} = \frac{6}{9} - \frac{18}{9} - \frac{4}{9} = -\frac{16}{9} \Rightarrow y'' = -\frac{16}{27}$.

Key idea: $y'' < 0$ so curve is <u>concave down</u> near P. Sketch should show curve below tangent near P.

Continued on page 6

[5] **4.** Consider the behaviour near $x = 0$ for the function

$$f(x) = \ln(1 + \sin(7x)). \qquad \text{[Note: } \ln = \log_e.\text{]}$$

Simply typing this formula into a modern computer leads to the values in the following table, where f_c denotes the computer's approximation to the true function f.

x	5.000×10^{-15}	5.000×10^{-16}	5.000×10^{-17}	5.000×10^{-18}
$f_c(x)$	3.508×10^{-14}	3.553×10^{-15}	4.441×10^{-16}	0.000
$f(x)$	3.500×10^{-14}	3.500×10^{-15}	3.500×10^{-16}	3.500×10^{-17}

The computed values are not very accurate, and they get worse as the input x approaches 0. (All computers suffer from "roundoff error". Handheld calculators are typically even less accurate.)

Use a suitable tangent-line approximation to generate accurate values and fill in the bottom line of the table. Report the same number of significant figures shown for $f_c(x)$.

$$f'(x) = \frac{1}{1 + \sin(7x)}\left[\cos(7x) \cdot 7\right]$$

$\therefore$ $f'(0) = 7$, so tan line for $y = f(x)$ at point $(x, y) = (0, 0)$ has equation

$$y = 0 + 7(x - 0) = 7x.$$

Using tan-line values instead of exact f-values gives correct results in table.

Continued on page 7

[5] **5.** Ocean water absorbs sunlight, so that the light intensity $L(x)$ at depth x below the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL$$

for some constant k. Experienced divers in the waters off Haida Gwaii know that at a depth of 6 m, the light intensity is half its value at the surface. They can work without artificial light down to a depth where the light intensity is one-tenth of its value at the surface. How deep is this?

Recognize that equation, know it forces

$$L(x) = Ae^{-kx}$$

for some constant A. Interpret $A = L(0)$ as light intensity at surface, so divers know

$$\tfrac{1}{2}A = L(6) = Ae^{-6k}.$$

Get $\ln\left(\frac{1}{2}\right) = \ln\left(e^{-6k}\right) = -6k$, so $k = \frac{1}{6}\ln 2$.

We'll have $\frac{1}{10}A = Ae^{-kx}$ when

$$\frac{1}{10} = e^{-kx}, \text{ i.e., } -\ln 10 = -kx:$$

depth $x = \frac{1}{k}\ln(10) = \dfrac{6\ln(10)}{\ln(2)}$. $(\approx 19.93 \text{ m})$

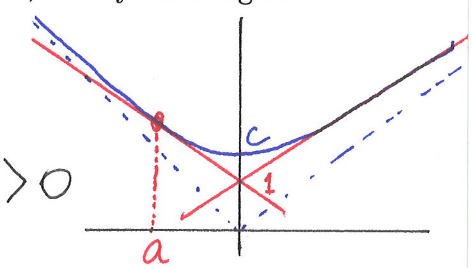
Continued on page 8

[8] **6.** Consider the curve $y = \sqrt{c^2 + x^2}$, where c is a constant obeying $c > 1$.

(a) Show that the curve is concave up and make a rough sketch, clearly labelling all intercepts.

$$y' = \frac{\cancel{2}x}{\cancel{2}\sqrt{c^2+x^2}} \Rightarrow y'' = \frac{\sqrt{c^2+x^2} - x\left(\frac{x}{\sqrt{c^2+x^2}}\right)}{c^2+x^2}$$

$$= \frac{(c^2+x^2)-x^2}{(c^2+x^2)^{3/2}} = \frac{c^2}{(c^2+x^2)^{3/2}} > 0$$



Since $y'' > 0$ always, Curve is concave up.

(b) Find the (x, y)-coordinates of each point on the curve from which the tangent line passes through the point $(0, 1)$.

Let unknown a be x-coord for tangency.

Tan line: $y = \sqrt{c^2+a^2} + \dfrac{a}{\sqrt{c^2+a^2}}(x-a)$.

Line hits pt. $(0,1)$ if and only if a satisfies

$$1 = \sqrt{c^2+a^2} + \frac{a(-a)}{\sqrt{c^2+a^2}} = \frac{(c^2+a^2) - a^2}{\sqrt{c^2+a^2}}$$

$$\Leftrightarrow \quad \sqrt{c^2+a^2} = c^2$$

$$\Leftrightarrow \quad c^2+a^2 = c^4 \quad (*)$$

$$\Leftrightarrow \quad a^2 = c^4 - c^2 = c^2(c^2-1)$$

So $a = \pm c\sqrt{c^2-1}$ give 2 pts of tangency, where

$$y = \sqrt{c^2+a^2} \overset{(*)}{=} \sqrt{c^4} = c^2.$$

Summary: Desired pts $(x,y) = \left(\pm c\sqrt{c^2-1},\ c^2\right)$.

Continued on page 9

[8] **7.** (a) Write the limit-based definition for the derivative $f'(a)$ associated with a given function f and point a.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \left[\text{OR} \quad \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right]$$

(b) Use limit-evaluation methods and the definition in part (a) to calculate $f'(1)$ for the function

$$f(x) = \frac{1}{x+3}.$$

[Do not use differentiation rules in this part.]

$$f'(1) = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(1+h)+3} - \frac{1}{1+3}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{4-(4+h)}{(4+h)(4)}\right]$$

$$= \lim_{h\to 0} \frac{-1}{(4+h)(4)} = -\frac{1}{16}.$$

(c) Find $L = \lim_{x\to 0} (1+\tan(3x))^{1/x}$ by matching $\ln(L)$ with the definition in part (a). [Differentiation rules are welcome in this part.]

$$\ln(L) = \lim_{x\to 0} \ln\left((1+\tan 3x)^{1/x}\right) = \lim_{x\to 0} \frac{\ln(1+\tan(3x))}{x}$$

$$= f'(0), \text{ for } f(x) = \ln(1+\tan(3x)).$$

[Notice $f(0) = \ln(1) = 0$.] Calc $f'(x) = \dfrac{\sec^2(3x)\cdot(3)}{1+\tan(3x)}$, so

$f'(0) = 3$. Now $\ln(L) = 3$ gives

$$\boxed{L = e^3.}$$

Continued on page 10

[6] **8.** Consider this proposed identity:

$$\frac{d}{dx}\left(y^2\right) = \left(\frac{dy}{dx}\right)^2, \qquad \text{for all real } x. \qquad (*)$$

[Throughout this question, consider only $y = f(x)$ such that f' is continuous on $\mathbb{R}$.]

(a) Find **one function** $y = f(x)$ for which statement $(*)$ is **false**.

$y=x$ will do: $\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) = 2x$

but $\left(\frac{dy}{dx}\right)^2 = (1)^2 = 1$

These are different!

(b) Find **all functions** $y = f(x)$ for which statement $(*)$ is **true**.

Expand the left side: $(*) \Leftrightarrow \quad 2yy' = (y')^2$,

i.e., $\quad (y')\left[y' - 2y\right] = 0.$

Case 1: $y' = 0$.
This holds if and only if y is constant.

Case 2: $y' - 2y = 0$.
This holds if and only if $y = Ae^{2x}$ for some constant A.

Case 3: $y'(x) = 0$ for some x, $y'(x) = 2y(x)$ for all other x.
This is incompatible with continuity of y':
cases 1-2 are exhaustive.

SUMMARY: All constant multiples of the constant 1 and the fcn e^{2x} obey $(*)$. No other smooth solutions exist.

Continued on page 11

[6] **9.** The acceleration of an aircraft t seconds after it starts its take-off run is $2 + \dfrac{t}{5}$ meters/sec^2. If the aircraft is not moving at $t = 0$, and it will take off when its speed reaches 30 meters/sec, what distance will it travel before it takes off?

Acceleration $a = 2 + t/5$ equals $\dfrac{dv}{dt}$ (v being velocity) so $v = 2t + \dfrac{t^2}{10} + C$ for some const C. Given $0 = v(0) = C$, get

$$v(t) = 2t + t^2/10 .$$

Take-off time T is defined by

$$30 = v(T) = 2T + T^2/10 ,$$ i.e.,

$$0 = T^2 + 20T - 300 = (T-10)(T+30)$$ so $T = 10$ or $T = -30$.

Key word "after" rejects $T = -30$, so $T = 10$.

Now $v = 2t + \dfrac{t^2}{10}$ equals $\dfrac{dx}{dt}$ (x being distance from start of take-off roll), so

$$x = t^2 + \frac{t^3}{30} + K$$

for some const K. Use $0 = x(0) = K$ to find distance requested:

$$x(T) = x(10) = 100 + \frac{1000}{30} = 100\left(\frac{4}{3}\right) = \frac{400}{3} \text{ (m)}.$$

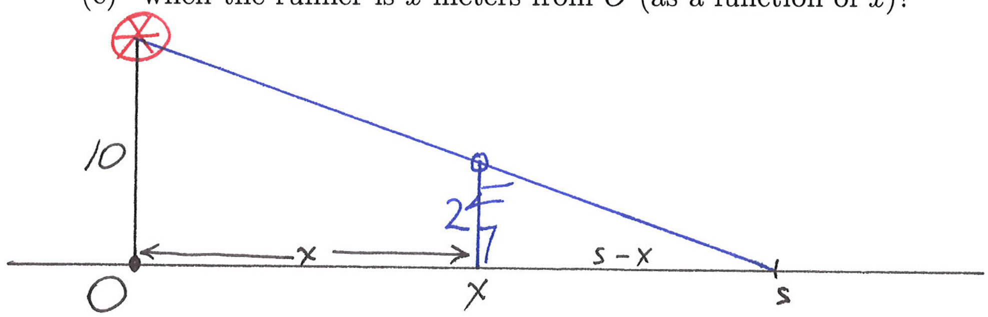
Continued on page 12

[8] **10.** A fugitive whose height is 2 meters runs straight away from a searchlight mounted 10 meters above a point O on the ground. The gound is horizontal; the runner's speed is 8 meters per second. How fast is the shadow of the runner's head moving along the ground . . .

(a) when the runner is 15 meters from O?

(b) when the runner is 25 meters from O?

(c) when the runner is x meters from O (as a function of x)?



Take x as described in the question; let s be the distance from O to the head's shadow.

Similar Δ's : $\frac{s}{10} = \frac{s-x}{2}$

$\iff \quad s = 5s - 5x$

$\iff \quad 5x = 4s$ ... identity valid for all t.

Given $\frac{dx}{dt} = 8$ m/s, deduce $5\frac{dx}{dt} = 4\frac{ds}{dt}$

$\Rightarrow \frac{ds}{dt} = \frac{5}{4}(8 \text{ m/s}) = 10$ m/s.

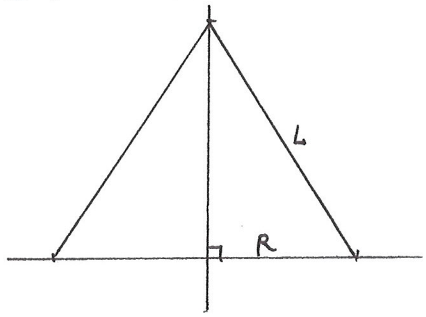
All of (a)(b)(c) have the same answer: 10 m/s, indep of x.

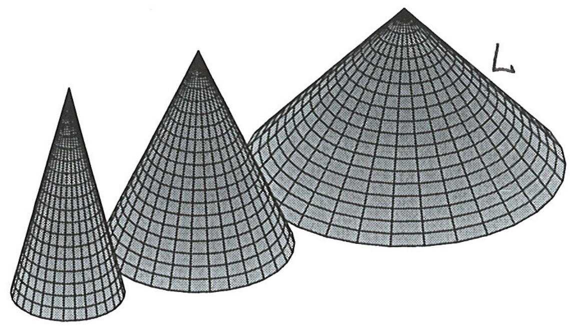
Continued on page 13

[9] **11.** There are infinitely many right circular cones with a slant height of $L = 3$ metres. Find the base radius R for the one with the largest volume. (The "slant height" of a cone is the length of a line from its vertex to a point on the perimeter of its circular base. Many such lines are visible in the sketch below.)

Sample Cones of Slant Height L=3

SIDE VIEW





Famous cone formula $V = \frac{1}{3}\pi R^2 h$, where h is $\perp$ height.

Pythagoras: $h^2 + R^2 = L^2 \Rightarrow h = \sqrt{L^2 - R^2}$, so

$$V(R) = \frac{\pi}{3} R^2 \sqrt{L^2 - R^2}.$$

Maximizers for V will be maximizers for $f = \left(\frac{3}{\pi} V\right)^2$, i.e.,

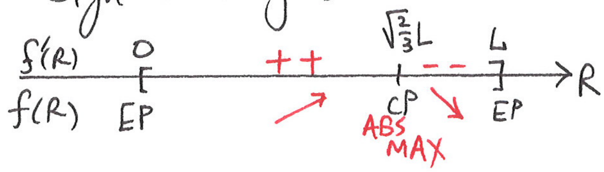
$$f(R) = R^4 (L^2 - R^2).$$

Now $f'(R) = 4R^3(L^2 - R^2) + R^4(-2R) = R^3\left[4L^2 - 4R^2 - 2R^2\right]$

$$= 2R^3(2L^2 - 3R^2).$$

CPs: $R = \pm\sqrt{\frac{2}{3}}\, L$

Sign analysis on natural domain $0 \le R \le L$:



MAX occurs when $R = \sqrt{\frac{2}{3}}\, L$. Setup $L = 3$m gives $R = \sqrt{6}$ m.

Continued on page 14

[9] **12.** Using the axes provided on the next page, make a reasonable sketch of the curve $y = f(x)$, using the information below:

$$\lim_{x\to-\infty} f(x) = 1, \quad f(-1) = 0, \quad f(0) = 1, \quad f(1) = 2, \quad \lim_{x\to+\infty} f(x) = 1,$$

$$f'(x) = \frac{2(1-x^2)}{(1+x^2)^2} \text{ for all real numbers } x.$$

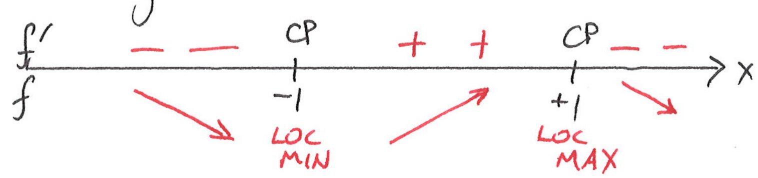
Support your sketch with calculations that identify the following features:

(a) Exact intervals on which the curve is increasing or decreasing, and x-coordinates for any local maximum or minimum points.

(b) Exact intervals on which the curve is concave up or concave down, and x-coordinates for any inflection points.

[Note: The formula given above is for f', not for f. A formula for f is not needed to complete this question.]

(a) Sign analysis for $f'(x)$: sign matches that of $1-x^2$, with changes when $1-x^2=0$, i.e., $x=\pm 1$.

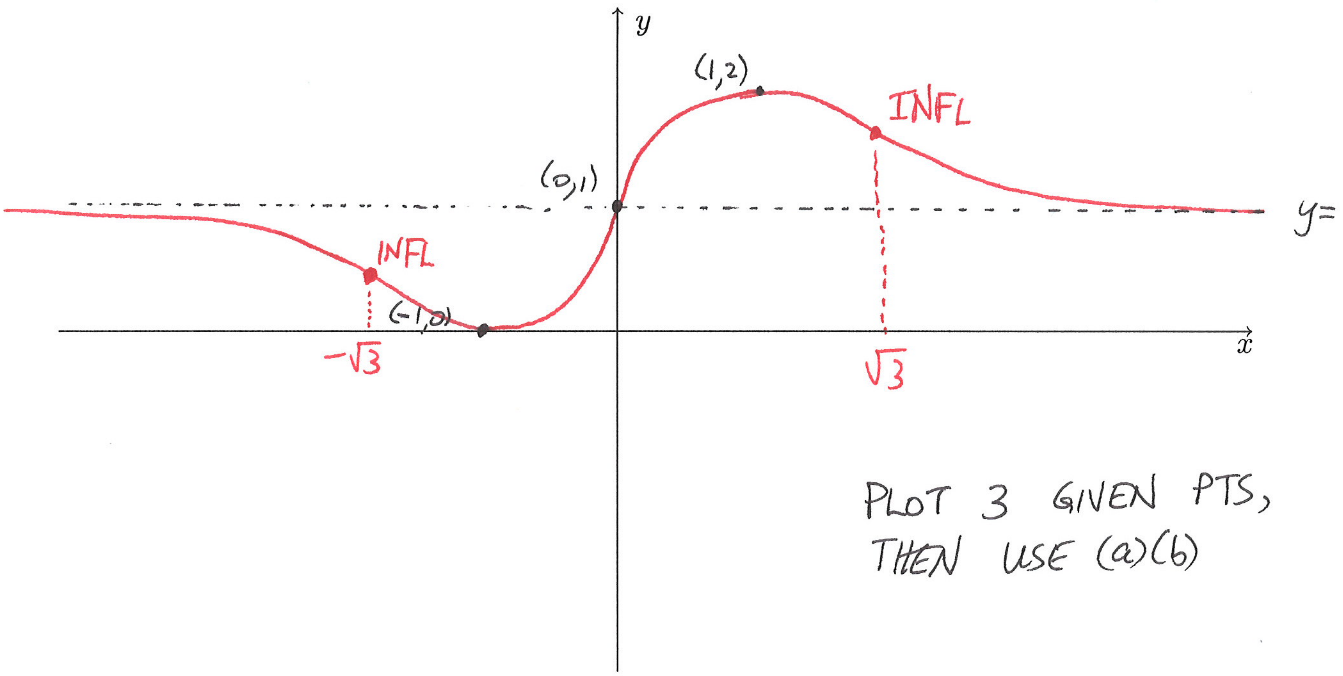


Curve increasing on interval $[-1, 1]$

Curve decreasing on int's $(-\infty, -1]$ and $[+1, +\infty)$

(but NOT on set $(-\infty,-1] \cup [+1,+\infty)$!)

Continued on page 15

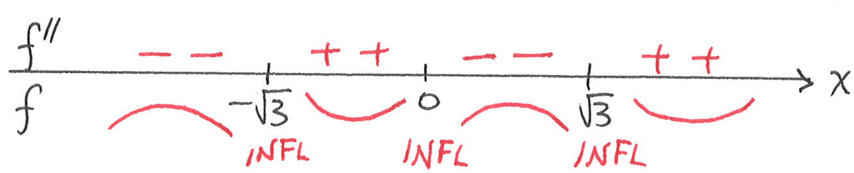


PLOT 3 GIVEN PTS, THEN USE (a)(b)

(b) $$f''(x) = \frac{d}{dx}\left(f'(x)\right) = 2 \cdot \frac{-2x(1+x^2)^2 - (1-x^2)\,2(1+x^2)[2x]}{(1+x^2)^4}$$

$$= 2\,\frac{-2x(1+x^2) - 4x(1-x^2)}{(1+x^2)^3} = 4x \cdot \frac{x^2-3}{(1+x^2)^3}.$$

Sign of $f''(x)$ matches sign of $x(x^2-3)$, with simple changes when $x=0, \pm\sqrt{3}$. Those are inflection pts.



Curve concave up on intervals $[-\sqrt{3},0]$ and $[\sqrt{3},+\infty)$ (separately)

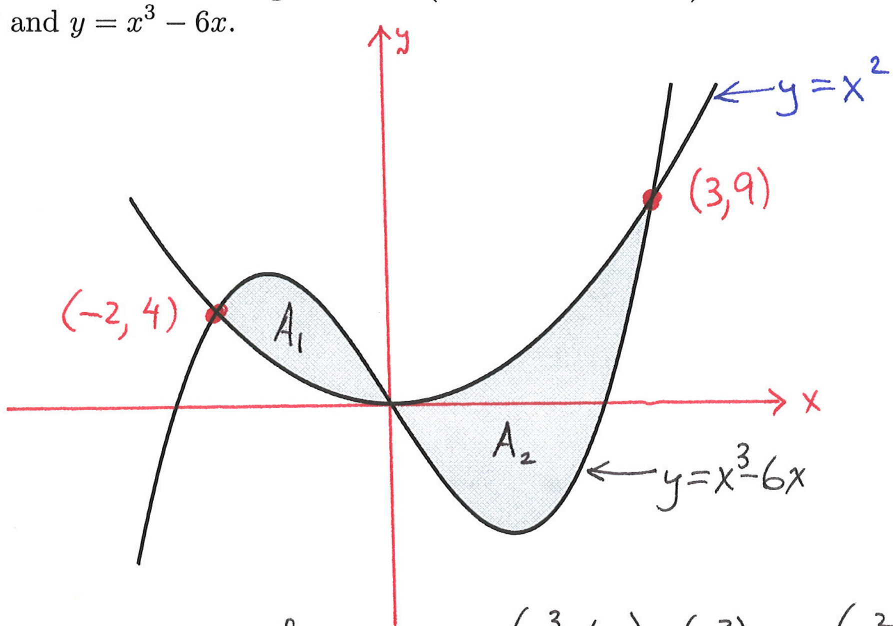
Curve concave down on intervals $(-\infty,-\sqrt{3}]$ and $[0,\sqrt{3}]$ (separately)

NOTES: (1) $f'(0) \simeq 2$ approximately correct in sketch

(2) $f'(x)$ even $\Rightarrow$ $f(x)+C$ odd for some C. Continued on page 16

Sketch shows $C=-1$ ... odd symm a useful check.

[7] **13.** Find the shaded area in the figure below (not drawn to scale). The curves involved are $y = x^2$ and $y = x^3 - 6x$.



Given curves cross when $0 = (x^3-6x)-(x^2) = x(x^2-x-6)$
$= x(x-3)(x+2)$.

This lets us add axes & labels to sketch as above.

Add two contributions: $A = A_1 + A_2$, where

$$A_1 = \int_{-2}^{0}\left[(x^3-6x)-x^2\right]dx = \left[\frac{1}{4}x^4 - 3x^2 - \frac{1}{3}x^3\right]_{-2}^{0} = 0 - \left[4 - 12 + \frac{8}{3}\right]$$

$$A_2 = \int_0^3 \left[x^2-(x^3-6x)\right]dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 + 3x^2\right]_0^3 = \left[9 - \frac{81}{4} + 27\right] - 0$$

Summary: $A = \left(8 - \frac{8}{3}\right) + \left(36 - \frac{81}{4}\right) = 44 - \left(\frac{8}{3} + \frac{81}{4}\right) = 21 + \frac{1}{12}$

$(\simeq 21.0833)$

The End