This examination has 16 pages including this cover.

UBC-SFU-UVic-UNBC Calculus Examination

9 June 2011, 12:00-15:00 PDT

Name: ______________________ **Signature:** ______________________

School: ______________________ **Candidate Number:** ______________________

Rules and Instructions

1. *Show all your work!* Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. Some basic formulas appear on page 2. No other notes, books, or aids are allowed. In particular, *all calculator memories must be empty when the exam begins.*
5. If you need more space to solve a problem on page n, work on the back of page $n - 1$.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

1		10
2		9
3		10
4		5
5		5
6		8
7		8
8		6
9		6
10		8
11		9
12		9
13		7
Total		100

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Formula Sheet for 9 June 2011

Exact Values of Trigonometric Functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Trigonometric Definitions and Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Continued on page 3

[10] **1.** (a) Find $f'(x)$, but do not simplify, given $f(x) = \dfrac{\sin(x)\cos(x)}{1 + e^{3x}}$.

(b) Find $\dfrac{dy}{dx}$, but do not simplify, given $y = \sin\left(\sqrt{1 + x^4} - x^2\right)$.

(c) Find $\displaystyle\int \frac{dx}{1 + 25x}$.

(d) Find $\displaystyle\int \frac{dx}{1 + 25x^2}$.

Continued on page 4

[9] **2.** Find the exact values of these limits. Show your work.

(a) $\displaystyle\lim_{x\to 2} \frac{x^3-8}{x^2-4}$.

(b) $\displaystyle\lim_{x\to 0} \frac{x}{\sqrt{1+7x}-1}$.

(c) $\displaystyle\lim_{x\to \infty} \frac{a^x}{1+a^x}$, where $a > 0$ is constant. (Explain how the answer depends on a.)

Continued on page 5

[10] **3.** Both parts of this question concern the point P with coordinates $(1,2)$ on the curve

$$x^3 + y^3 = 3xy + 3. \qquad (*)$$

(a) Sketch and find the area of the triangle described as follows. One vertex is at the origin. The other two vertices are the x and y intercepts of the line that is tangent to the curve in $(*)$ at the point P.

(b) Calculate y'' at the point P for the curve in $(*)$. Use your answer to make a rough sketch that shows the relationship of the curve to its tangent line near P. (If you can combine this sketch with the one requested in part (a), please do so.)

Continued on page 6

[5] **4.** Consider the behaviour near $x = 0$ for the function

$$f(x) = \ln(1 + \sin(7x)). \qquad \text{[Note: } \ln = \log_e\text{.]}$$

Simply typing this formula into a modern computer leads to the values in the following table, where f_c denotes the computer's approximation to the true function f.

x	5.000×10^{-15}	5.000×10^{-16}	5.000×10^{-17}	5.000×10^{-18}
$f_c(x)$	3.508×10^{-14}	3.553×10^{-15}	4.441×10^{-16}	0.000
$f(x)$				

The computed values are not very accurate, and they get worse as the input x approaches 0. (All computers suffer from "roundoff error". Handheld calculators are typically even less accurate.)

Use a suitable tangent-line approximation to generate accurate values and fill in the bottom line of the table. Report the same number of significant figures shown for $f_c(x)$.

Continued on page 7

[5] **5.** Ocean water absorbs sunlight, so that the light intensity $L(x)$ at depth x below the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL$$

for some constant k. Experienced divers in the waters off Haida Gwaii know that at a depth of 6 m, the light intensity is half its value at the surface. They can work without artificial light down to a depth where the light intensity is one-tenth of its value at the surface. How deep is this?

Continued on page 8

[8] **6.** Consider the curve $y = \sqrt{c^2 + x^2}$, where c is a constant obeying $c > 1$.

(a) Show that the curve is concave up and make a rough sketch, clearly labelling all intercepts.

(b) Find the (x, y)-coordinates of each point on the curve from which the tangent line passes through the point $(0, 1)$.

Continued on page 9

[8] **7.** (a) Write the limit-based definition for the derivative $f'(a)$ associated with a given function f and point a.

(b) Use limit-evaluation methods and the definition in part (a) to calculate $f'(1)$ for the function

$$f(x) = \frac{1}{x+3}.$$

[Do not use differentiation rules in this part.]

(c) Find $L = \lim_{x \to 0} (1 + \tan(3x))^{1/x}$ by matching $\ln(L)$ with the definition in part (a). [Differentiation rules are welcome in this part.]

Continued on page 10

[6] **8.** Consider this proposed identity:

$$\frac{d}{dx}\left(y^2\right) = \left(\frac{dy}{dx}\right)^2, \qquad \text{for all real } x. \qquad (*)$$

[Throughout this question, consider only $y = f(x)$ such that f' is continuous on $\mathbb{R}$.]

(a) Find **one function** $y = f(x)$ for which statement $(*)$ is **false**.

(b) Find **all functions** $y = f(x)$ for which statement $(*)$ is **true**.

Continued on page 11

[6] **9.** The acceleration of an aircraft t seconds after it starts its take-off run is $2 + \dfrac{t}{5}$ meters/sec^2. If the aircraft is not moving at $t = 0$, and it will take off when its speed reaches 30 meters/sec, what distance will it travel before it takes off?

Continued on page 12

[8] **10.** A fugitive whose height is 2 meters runs straight away from a searchlight mounted 10 meters above a point O on the ground. The gound is horizontal; the runner's speed is 8 meters per second. How fast is the shadow of the runner's head moving along the ground . . .

(a) when the runner is 15 meters from O?

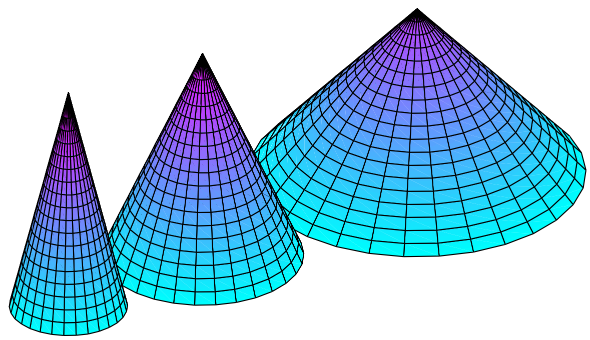
(b) when the runner is 25 meters from O?

(c) when the runner is x meters from O (as a function of x)?

Continued on page 13

[9] **11.** There are infinitely many right circular cones with a slant height of $L = 3$ metres. Find the base radius R for the one with the largest volume. (The "slant height" of a cone is the length of a line from its vertex to a point on the perimeter of its circular base. Many such lines are visible in the sketch below.)

Sample Cones of Slant Height L=3



Continued on page 14

[9] **12.** Using the axes provided on the next page, make a reasonable sketch of the curve $y = f(x)$, using the information below:

$$\lim_{x\to-\infty} f(x) = 1, \quad f(-1) = 0, \quad f(0) = 1, \quad f(1) = 2, \quad \lim_{x\to+\infty} f(x) = 1,$$

$$f'(x) = \frac{2(1-x^2)}{(1+x^2)^2} \text{ for all real numbers } x.$$

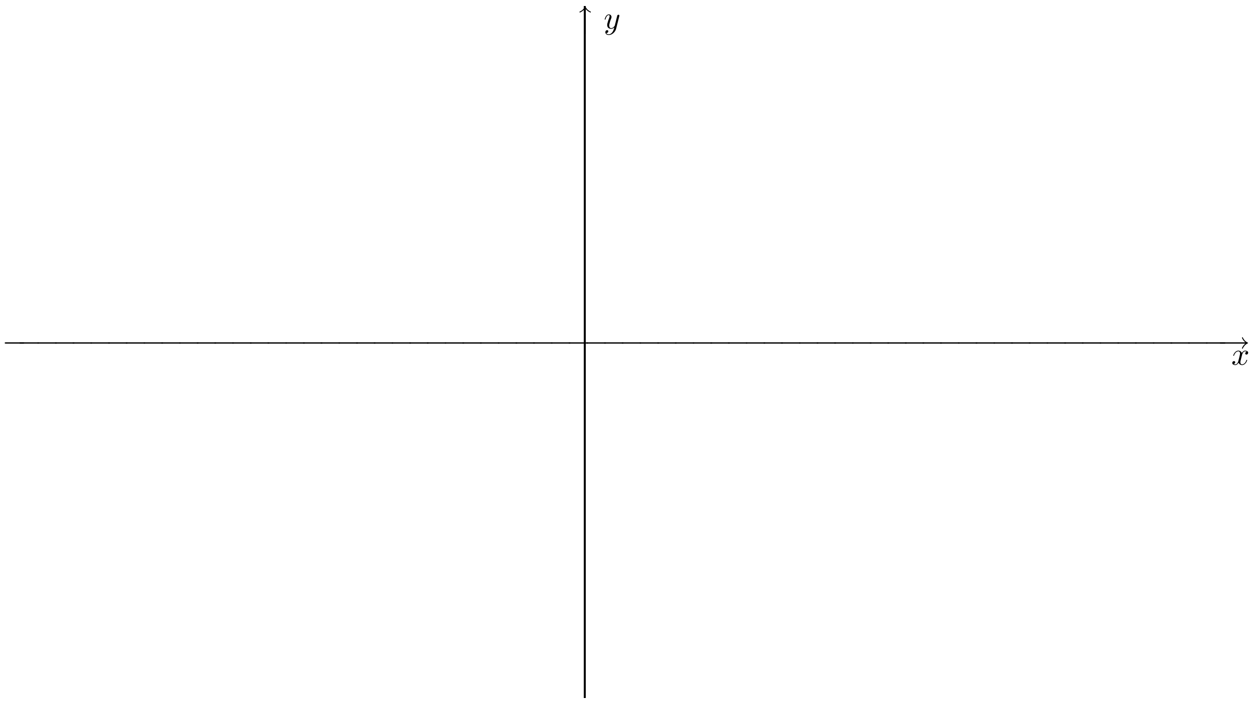
Support your sketch with calculations that identify the following features:

(a) Exact intervals on which the curve is increasing or decreasing, and x-coordinates for any local maximum or minimum points.

(b) Exact intervals on which the curve is concave up or concave down, and x-coordinates for any inflection points.

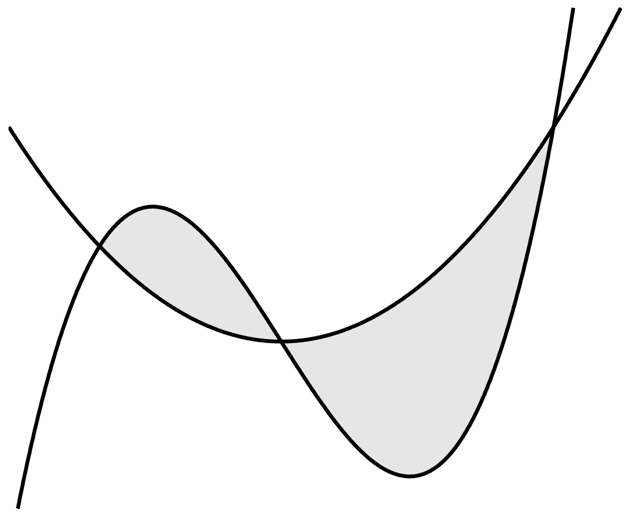
[Note: The formula given above is for f', not for f. A formula for f is not needed to complete this question.]

Continued on page 15



Continued on page 16

[7] **13.** Find the shaded area in the figure below (not drawn to scale). The curves involved are $y = x^2$ and $y = x^3 - 6x$.



The End