SFU-UBC-UNBC-UVic

Calculus Challenge Exam Solutions

June 10, 2010, 12:00-15:00

Name: ______________________________ (please print)
family name *given name*

School: ______________________________

Signature: ______________________________

Question	Maximum	Score
1	6	
2	8	
3	6	
4	6	
5	6	
6	6	
7	10	
8	8	
9	8	
10	6	
11	8	
12	8	
13	8	
14	6	
Total	100	

Instructions:

1. Show all your work. Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete.
2. A non-graphing, non-programmable calculator which meets ministry standards for the Provincial Examination in Principles of Mathematics 12 may be used. However, calculators are not needed. Correct answers that are calculator ready, like $3 + ln7$ or e^2 , are preferred.
3. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
4. If you need more space to solve a problem, use the back of the facing page.
5. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.

1. For each of the following evaluate the limit if it exists or otherwise explain why it does not exist.

[2] (a) $\lim\limits_{x\to 2} \dfrac{\frac{1}{x}-\frac{1}{2}}{x-2} = \lim\limits_{x\to 2} \dfrac{2-x}{2(2-x)} = \lim\limits_{x\to 2} -\dfrac{1}{2x} = -\dfrac{1}{4}$

[2] (b) $\lim\limits_{x\to -4^-} \dfrac{|x+4|}{x+4} = \lim\limits_{x\to -4^-} \dfrac{-(x+4)}{x+4} = -1$

[2] (c) $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{1+2x^2}} = \lim\limits_{x\to\infty} \dfrac{\frac{x}{x}}{\frac{\sqrt{1+2x^2}}{x}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{\frac{1}{x^2}+2}} = \dfrac{1}{\sqrt{2}}$

2. Differentiate each of the following with respect to x.

[2] (a) $y = e^{4x}$

$y' = e^{4x}(\frac{d}{dx}4x) = 4e^{4x}$

[2] (b) $y = \dfrac{3x-5}{x^2+1}$

$$y' = \frac{\left[\frac{d}{dx}(3x-5)\right](x^2+1) - \left[\frac{d}{dx}(x^2+1)\right](3x-5)}{(x^2+1)^2} = \frac{3(x^2+1) - 2x(3x-5)}{(x^2+1)^2} = \frac{-3x^2+10x+3}{(x^2+1)^2}$$

[2] (c) $y = x\ln(x^2+4)$

$$y' = (\frac{d}{dx}x)\ln(x^2+4) + x\cdot\frac{d}{dx}\ln(x^2+4) = \ln(x^2+4) + x\cdot\frac{2x}{x^2+4} = \ln(x^2+4) + \frac{2x^2}{x^2+4}$$

[2] (d) $y = \sin(x^2) - \sin^2(x)$

$$y' = \cos(x^2)\cdot\frac{d}{dx}(x^2) - 2\sin(x)\cdot\frac{d}{dx}\sin(x) = 2x\cos(x^2) - 2\sin(x)\cos(x)$$

[6] **3. Use the definition of derivative** to find $f'(x)$ where $f(x) = \sqrt{2x+1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} =$$

$$\lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}} =$$

$$\lim_{h\to 0} \frac{[2(x+h)+1] - (2x+1)}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)} = \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)} =$$

$$\lim_{h\to 0} \frac{2}{\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)} = \frac{1}{\sqrt{2x+1}}$$

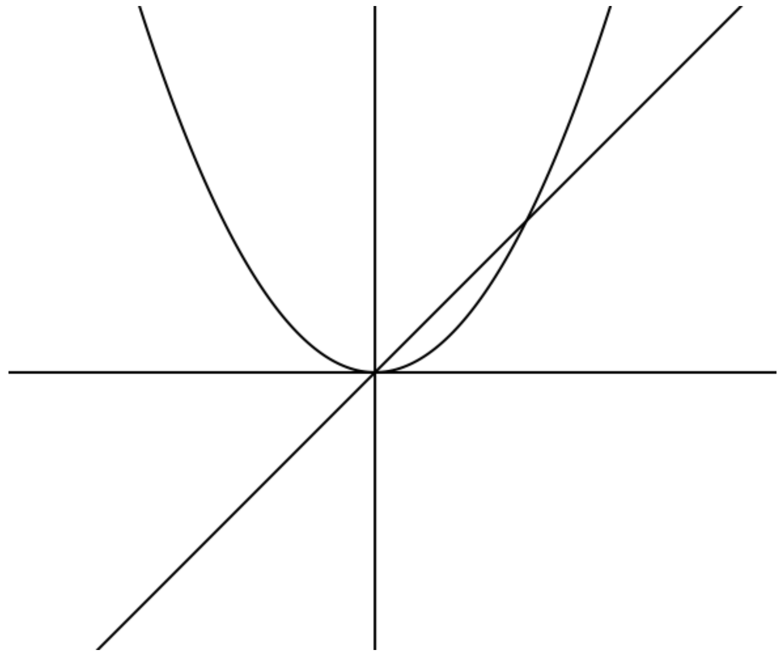
4. Evaluate the following antiderivatives.

[2] (a) $\displaystyle\int \left(\frac{1}{2}x^2 - 2x + 6\right)\,dx = \frac{1}{2}\cdot\frac{1}{3}x^3 - 2\cdot\frac{1}{2}x^2 + 6x + c = \frac{1}{6}x^3 - x^2 + 6x + c$

[2] (b) $\displaystyle\int (3e^x + 7)\,dx = 3e^x + 7x + c$

[2] (c) $\displaystyle\int \left(2\sqrt{x} + 6\cos x\right)\,dx = 2\cdot\frac{2}{3}x^{\frac{3}{2}} + 6\sin x + c = \frac{4}{3}x^{\frac{3}{2}} + 6\sin x + c$

[6] **5.** Find the area of the region bounded by the curves $y = x$ and $y = x^2$. **Sketch the graph.**



The area is

$$\int_0^1 \left(x - x^2\right)\, dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

6. In this question we investigate the solution of the equation

$$2x = \cos x.$$

[3] (a) Explain why you know the equation has **at least** one solution.

Note that the function $f(x) = 2x - \cos x$ is continuous on $(-\infty, \infty)$ we have

$$f(0) = -1 < 0$$

and

$$f(\frac{\pi}{2}) = \pi > 0$$

By the intermediate Value Theorem, it follows that $f(c) = 0$ for some $c \in (-1, \pi)$.

[3] (b) Use Newton's Method to approximate the solution of the equation by starting with $x_1 = 0$ and finding x_2.
(Note that you are being asked to find only one iteration of Newton's Method.)

In general $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{2x_n - \cos x_n}{2 + \sin x_n}$ and so $x_2 = 0 - \dfrac{2(0) - \cos(0)}{2 + \sin(0)} = \dfrac{1}{2}$

7. Let

$$f(x) = e^{1/x} \qquad f'(x) = -\frac{e^{1/x}}{x^2} \qquad f''(x) = \frac{e^{1/x}(2x+1)}{x^4}$$

[1] (a) What is the domain of f?

The domain is $(-\infty, 0) \cup (0, \infty)$.

[1] (b) Determine any points of intersection of the graph of f with the x and y axes.

Since $x = 0$ is not in the domain there is no y-intercept and since $e^{\frac{1}{x}} \neq 0$ for all x there are no x-intercepts.

[1] (c) Use limits to determine any horizontal asymptotes of f.

$\lim_{x \to \pm\infty} e^{\frac{1}{x}} = e^0 = 1$ and therefore $y = 1$ is a horizontal asymptote.

[1] (d) Use limits to determine any vertical asymptotes of f.

$\lim_{x \to 0^+} e^{\frac{1}{x}} = \infty$ and $\lim_{x \to 0^-} e^{\frac{1}{x}} = 0$. Therefore $x = 0$ is a vertical asymptote.

[1] (e) For each interval in the table below, indicate whether f is increasing or decreasing.

interval	$(-\infty, 0)$	$(0, \infty)$
$f(x)$	decreasing	decreasing

[1] (f) Determine the x coordinates of any local maximum or minimum values of f.

Since f is decreasing on $(-\infty, 0) \cup (0, \infty)$ there are no local extrema.

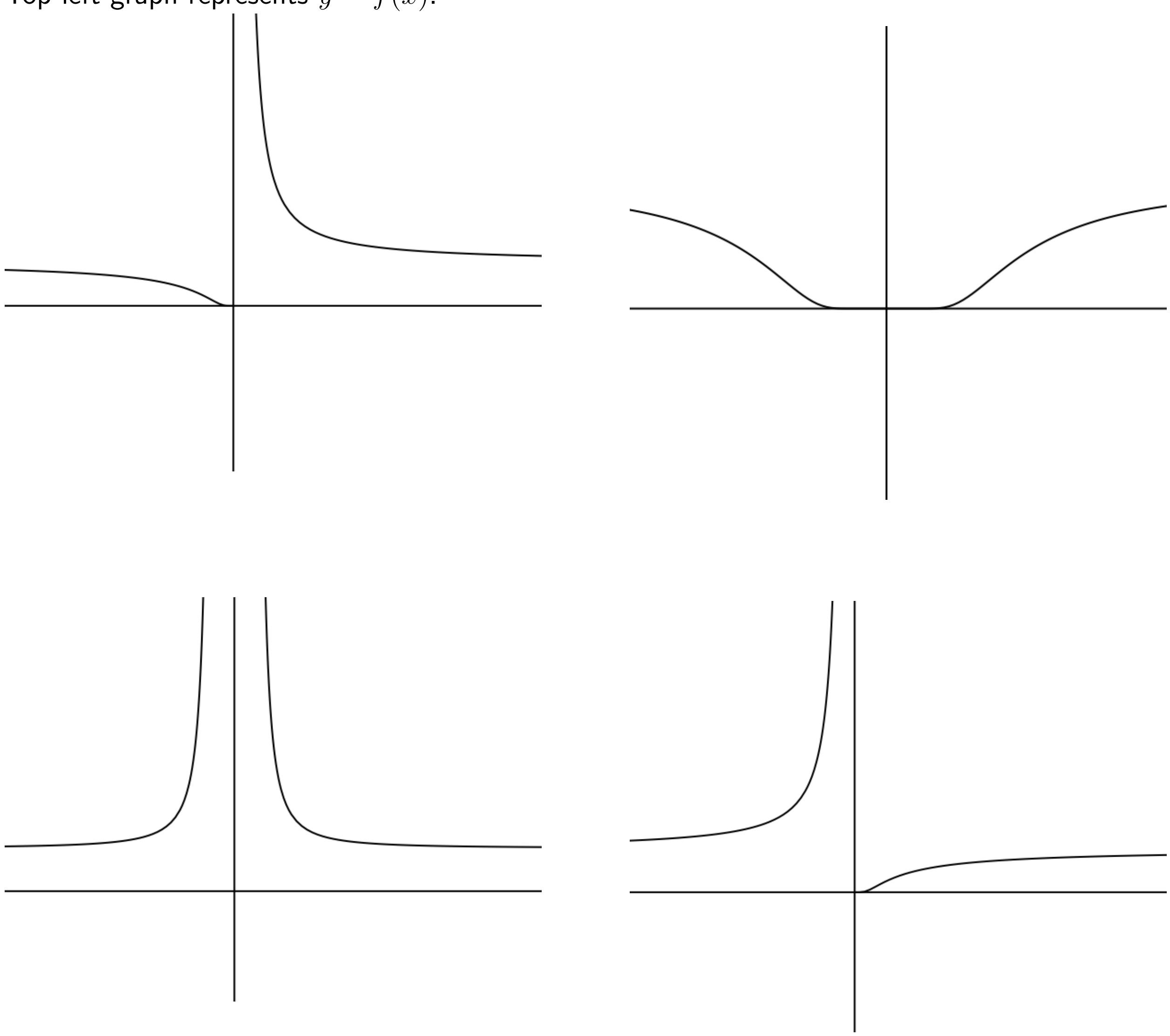
[1] (g) For each interval in the table below, indicate whether f is concave up or concave down.

interval	$(-\infty, -1/2)$	$(-1/2, 0)$	$(0, \infty)$
$f(x)$	concave down	concave up	concave up

[1] (h) Determine the x coordinates of any inflection points on the graph of f.

Since the concavity changes at $x = -\frac{1}{2}$ there is an inflection point there.

[2] (i) Which of the following best represents the graph of $y = f(x)$? Circle only one answer.
Top left graph represents $y = f(x)$.



[4] **8.** (a) Suppose that we do not have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2+5}$ for all x. Use a linear approximation to estimate $g(2.05)$.

A linear approximation gives

$$g(x + \Delta x) \approx g(x) + g'(x)\Delta x$$

and so

$$g(2.05) \approx g(2) + g'(2)(0.05) = -4 + 3(0.05) = -3.85$$

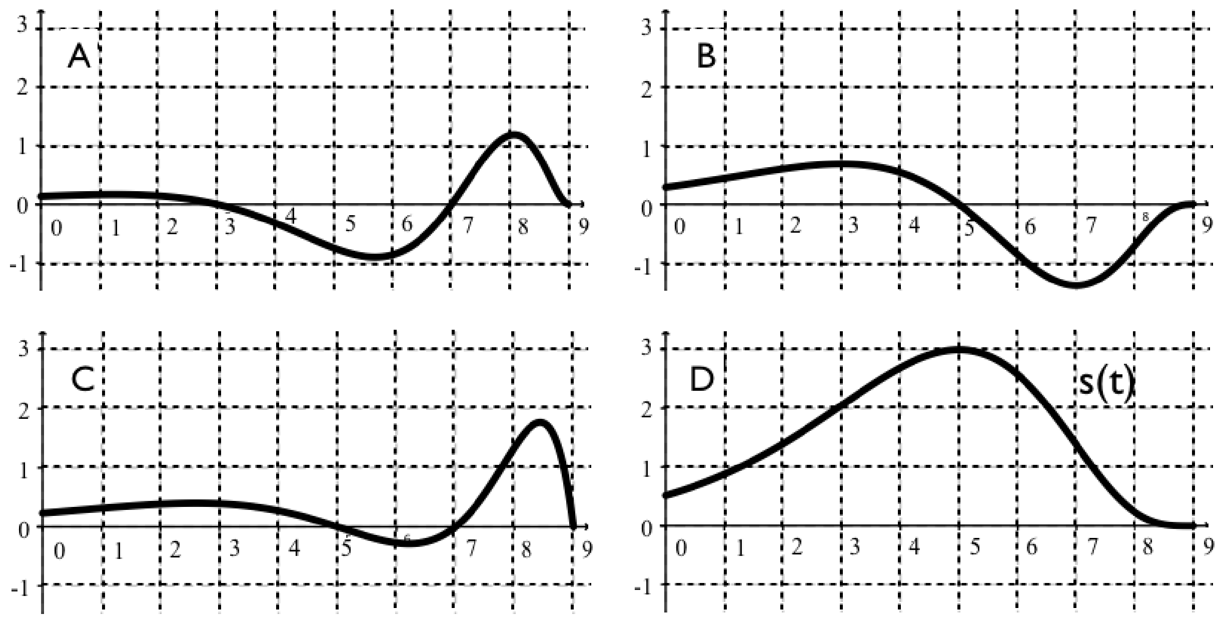
[4] (b) Is the estimate obtained in part (a) an overestimate or an underestimate of the actual value of $g(2.05)$? [Hint: Consider $g''(x)$.]

We have

$$g''(x) = \frac{d}{dx}\sqrt{x^2+5} = \frac{1}{2\sqrt{x^2+5}} \cdot 2x = \frac{x}{\sqrt{x^2+5}}$$

Since $g''(2) > 0$ the graph of g is concave up at $x = 2$. This tells us that the graph lies above the tangent line near $x = 2$ and so the linear approximation gives us an underestimate.

9. A particle moves along a line with a position function $s(t)$, where s is measured in meters and t in seconds. Four graphs are shown below: one corresponds to the function $s(t)$, one to the velocity $v(t)$ of the particle, one to its acceleration $a(t)$ and one is unrelated.



[3] (a) Identify the graphs of $s(t)$, $v(t)$ and $a(t)$ by writing the appropriate letter (A,B,C,D) in the space provided next to the function name. (The position function s is already labeled.)

$s =$ ___*D*___ , $v =$ ___*B*___, $a =$ ___*A*___

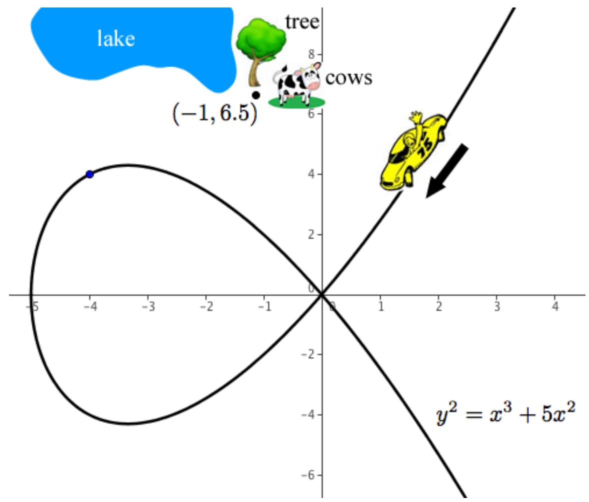
[4] (b) Find all time intervals when the particle is slowing down, and when it is speeding up. Justify your answer.

The particle is speeding up when v and a have the same sign which is on $(0, 3) \cup (5, 7)$. The particle is slowing down when v and a have opposite signs which is on $(3, 5) \cup (7, 9)$.

[1] (c) Find the total distance travelled by the particle over the interval $3 \leq t \leq 9$.

The total distance is $|s(5) - s(3)| + |s(9) - s(5)| = 1 + 3 = 4$ m.

10. A race car is speeding around a race-track and comes to a particularly dangerous curve in the shape $y^2 = x^3 + 5x^2$. The diagram below indicates the direction the car is traveling along the curve.



[2] (a) Find the derivative of y with respect to x.

Differentiating implicitly gives

$$2y \cdot \frac{dy}{dx} = 3x^2 + 10x$$
$$\frac{dy}{dx} = \frac{3x^2 + 10x}{2y}$$

[3] (b) If the car skids off at the point $(-4, 4)$ and continues in a straight path find the equation of the line the car will travel in.

The slope of the tangent line at $(-4, 4)$ is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(-4,4)} = \frac{3(-4)^2 + 10(-4)}{2(4)} = \frac{48 - 40}{8} = 1$$

Thus the equation of the tangent line is $y = x + 8$.

[1] (c) If a tree is located at the point $(-1, 6.5)$ with a lake to the left and cows to the right, will the car hit the lake, the tree or the cows?

Since the point $(-1, 7)$ lies on the tangent line the car would hit the lake.

11. A cup of coffee, cooling off in a room at temperature $20°$C, has cooling constant $k = 0.09\text{min}^{-1}$. Assume the temperature of the coffee obeys Newton's Law of Cooling.

[4] (a) Show that the temperature of the coffee is decreasing at a rate of $5.4°$C/min when its temperature is $T = 80°$C.

Newton's law of cooling is

$$-k(T - T_0) = \frac{dT}{dt}$$

where T_0 is the surrounding temperature. Thus when $T = 80$ we have

$$\frac{dT}{dt} = -0.09(80 - 20) = -5.4$$

[4] (b) The coffee is served at a temperature of $90°$. How long should you wait before drinking it if the optimal temperature is $65°$C? (It is preferred that you leave your answer in the exact form. i.e. as an expression that contains powers of e and/or logarithms.)

From $\frac{dT}{dt} = -k(T - 20)$ we have

$$\int \frac{dT}{T - 20} = \int -k\,dt$$

$$\ln|T - 20| = -kt + c_0$$

$$T - 20 = \pm e^{-kt+c_0} = c \cdot e^{-kt}$$

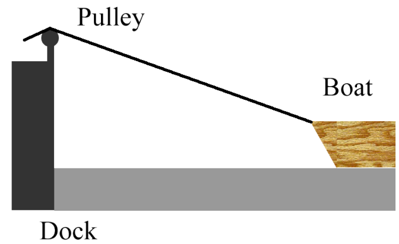
$$T = c \cdot e^{-kt} + 20$$

At $t = 0$ we have $T = 90$ and so $c = 70$. The time at which $T = 65$ is then

$$70 \cdot e^{-0.09t} + 20 = 65$$

$$t = \frac{\ln(\frac{45}{70})}{-0.09} = -\frac{100}{9} \cdot \ln(\frac{9}{14})$$

[8] **12.** A boat is pulled into a dock by means of a rope attached to a pulley on the dock. The rope is attached to the bow of the boat at a point 1 m below the pulley. If the rope is pulled through the pulley at a rate of 1 m/sec, at what rate will the boat be approaching the dock when there is 10 m of rope between the pulley and the boat?



Let x denote the distance from the boat to the dock and let z denote the length of the rope between the dock and the pulley at time t. Then

$$x^2 + 1 = z^2$$

$$\frac{d}{dt}(x^2 + 1) = \frac{d}{dt}z^2$$

$$2x \cdot \frac{dx}{dt} = 2z \cdot \frac{dz}{dt}$$

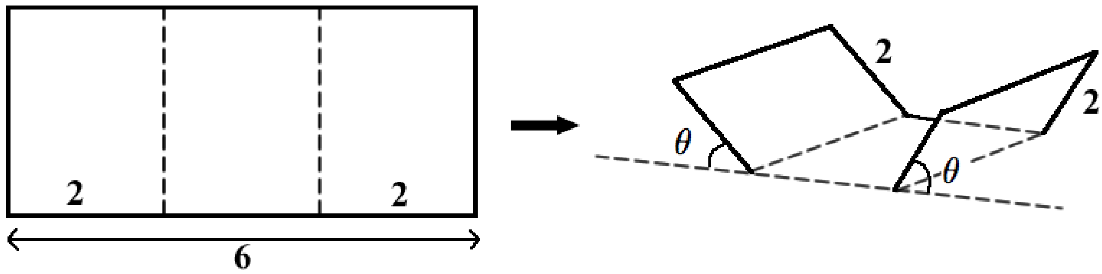
$$\frac{dx}{dt} = \frac{z}{x} \cdot \frac{dz}{dt}$$

When $z = 10$ we have $x = \sqrt{10^2 - 1} = \sqrt{99}$ and $\dfrac{dz}{dt} = -1$. Then

$$\frac{dx}{dt} = -\frac{10}{\sqrt{99}}$$

Therefore the boat will be approaching the dock at $\dfrac{10}{\sqrt{99}}$ m/s.

[8] **13.** A water trough is to be made from a long strip of tin 6 ft wide by bending up at an angle θ a 2 ft strip at each side. What angle θ would maximize the cross sectional area, and thus the volume, of the trough?



The cross sectional area is

$$A(\theta) = \frac{1}{2}\left[2 + (2 + 4\cos\theta)\right] \cdot 2\sin\theta = 4(1+\cos\theta)\sin\theta$$

where $0 \le \theta \le \frac{\pi}{2}$. Then we have

$$\begin{aligned} A'(\theta) &= 4\left[(-\sin\theta)\sin\theta + (1+\cos\theta)\cos\theta\right] \\ &= 4(\cos\theta + \cos^2\theta - \sin^2\theta) \\ &= 4(2\cos^2\theta + 2\cos\theta - 1) \\ &= 4(2\cos\theta - 1)(\cos\theta + 1) \end{aligned}$$

The critical points occur when $A'(\theta) = 0$ which is when $\theta = \frac{\pi}{3}$. Comparing the area at the critical point and the end points

$$A(\frac{\pi}{3}) = 3\sqrt{3} \qquad\qquad A(0) = 0 \qquad\qquad A(\frac{\pi}{2}) = 4$$

we can see that the cross sectional area is maximum when $\theta = \frac{\pi}{3}$.

[6] **14.** Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

Taking the antiderivative of $f'(x)$ we see that

$$f(x) = \frac{1}{4}x^4 + c$$

for some constant c.
Since $x+y=0$ has slope -1 we can use the derivative to determine the value of x where the tangent line touches the curve

$$f'(x) = -1$$
$$x^3 = -1$$
$$x = -1$$

From $x + y = 0$ we see that this corresponds to the point $(-1, 1)$. Thus

$$1 = \frac{1}{4}(-1)^4 + c$$
$$c = \frac{3}{4}$$

and so

$$f(x) = \frac{1}{4}x^4 + \frac{3}{4}$$