2009 Calculus Challenge Examination — Comments

[6] **1.** Well done. When errors were made, the most common mistakes were misquoting the quotient rule and/or forgetting to use the Chain Rule in part (b). Incidentally, everyone who skipped the quotient rule in favour of a combination of product rule with chain rule got this one right.

[6] **2.** Well done. Beyond the inevitable mechanical errors, some writers forgot to take the derivative of the inside function when dealing with the derivative of the natural logarithm. Another standard error is to find y' correctly, but to use its general form instead of its instantaneous value at $x = 1$ to define $t(x) = y'(x)x + b$ instead of $\ell(x) = y'(1)x + b$. Function t is not linear, so its graph is not a line of any kind ... in particular, a tangent line!

[5] **3.** Very well done by the strong majority of writers who understand implicit differentiation. It's easy to relax prematurely after differentiating the left side and just copy the right side unchanged, which amounts to the accidental assertion that the derivative of 11 is not 11. The few who tried to solve for y before taking derivatives did not make much progress and did not earn many marks.

[7] **4.** Poorly done. Most derivatives were incorrect, leaving little room for progress. Then students reached for Newton's Method instead of the tangent-line approximation. The idea of using a tangent line to estimate the value of a function near a known point did not come through clearly in many submissions. The statement of part (b) was apparently confusing: common answers included, "Since we are adding $\pi/10^{10}$ to the argument, the answer is larger than the exact answer," and "My approximation is obviously larger than 1."

[6] **5.** (a) Most students used l'Hospital's Rule (successfully); many added the fractions. Only a handful tried to make the limit look like a derivative (and of these, only one writer made it work).

(b) Most writers did not know how to start. Of the minority who tried the conjugate, most finished successfully. Dividing top and bottom by the fastest-growing denominator term was the most popular method, but l'Hospital's Rule also made an appearance.

[6] **6.** Few students made substantial progress. Most got stuck on the derivative. Many calculated a few terms, plotted, and declared a_3 the winner.

Of those who found the critical point $x = e$ correctly, most either forgot to justify that this point gives a maximum, or to choose between a_2 and a_3 instead of "a_e", or both.

An alternate approach that avoids "logarithmic differentiation" is to start with the identity

$$f(x) = x^{1/x} = \left(e^{\ln x}\right)^{1/x} = e^{x^{-1}\ln(x)}.$$

No writers took this approach.

[8] **7.** (a) Most students established the derivative/antiderivative relations between a, v, and x. Some forgot the integration constants when solving for v or x.

Many were confused about the turn-around time, solving $a(t) = 0$ or even $x(t) = 0$. Even among those who solved $v(t) = 0$, many did not go on to find the required value of $x(4)$.

(b) It was quite common to calculate net displacement instead of total distance travelled, but even students with not much else correct did write $s = \int_0^6 |v(t)|\,dt$.

[9] **8.** (a) Explanations of the form "Nothing is ever perfectly accurate in real life" were rewarded with a sage philosophical nod, but no marks. Some words related to some kind of calculus concept were expected. [1 mark]

(b) A number of writers cited evaporation in the context of Newton's Law of Cooling. It doesn't apply, but students were not penalized. [2 marks]

(c) The most spectacular calculator-ready answer to earn full marks was

$$T = \sqrt{\frac{50^{22}}{48^{20}}} + 20.$$

[6] **9.** A strong majority of writers scored 0 on this problem by writing an incorrect formula for $f'(x)$ and making deductions from it. A correct one-line formula, based on

$$\frac{d}{dx} = x/|x| \qquad \text{(note that this is undefined when } x = 0\text{)}$$

is

$$f'(x) = \sin(|x|) + x\cos(|x|)\frac{x}{|x|}.$$

This splits into a piecewise function that successful writers obtained by writing f in piecewise form from the beginning:

$$f(x) = \begin{cases} x\sin(x), & \text{if } x \geq 0, \\ -x\sin(x), & \text{if } x < 0, \end{cases} \implies f'(x) = \begin{cases} \sin(x) + x\cos(x), & \text{if } x > 0, \\ \text{DNE}, & \text{if } x = 0, \\ -\sin(x) - x\cos(x), & \text{if } x < 0; \end{cases}$$

$$\implies f''(x) = \begin{cases} 2\cos(x) - x\sin(x), & \text{if } x > 0, \\ \text{DNE}, & \text{if } x = 0, \\ -2\cos(x) + x\sin(x), & \text{if } x < 0. \end{cases}$$

The one-sided limits of $f'(x)$ as $x \to 0$ agree on the value 0, and this implies that the limit defining $f'(0)$ exists and shares this value. The one-sided limits of $f''(x)$ as $x \to 0$ disagree, and this implies that the one-sided limits in the definition of $f''(0)$ also disagree. These assertions are not trivial (they follow from L'Hospital's Rule), but they were implicitly used by the few students who managed this question correctly.

[9] **10.** The distance from O to Q at the instant of interest is $4\sqrt{2}$, and finding this was worth 2 marks. The markers were surprised by how few students took the time to do this.

Many successful alternatives exist. One involves Pythagoras a little more: writing y for the perpendicular distance from point P to segment $\overline{OQ}$ gives three identities valid for all time.

$$y = \sin\theta, \qquad u^2 + y^2 = 1, \qquad y^2 + w^2 = 5.$$

Therefore

$$\dot{y} = (\cos\theta)\dot{\theta}, \qquad u\dot{u} + y\dot{y} = 0, \qquad y\dot{y} + w\dot{w} = 0.$$

Combining $\dot{\theta} = 20\pi$ (given), with the instantaneous values $u = 1/\sqrt{2}$, $y = 1/\sqrt{2}$,$w = 7/\sqrt{2}$, allows various ways to solve for $\frac{d}{dt}\left|\overline{OQ}\right| = \dot{u} + \dot{w}$.

A very small minority let $z = \left|\overline{OQ}\right|$ and applied the cosine law to get

$$5^2 = 1^2 + z^2 - 2(1)(z)\cos\theta,$$

leading rapidly to

$$0 = z\dot{z} - \dot{z}\cos\theta + z(\sin\theta)\dot{\theta}.$$

Knowing $\theta = \pi/4$, $z = 4\sqrt{2}$, and $\dot{\theta} = 20\pi$ at the instant of interest gives the result very efficiently.

The most impressive "calculator-ready" answer to earn full marks was

$$\frac{d(\overline{OQ})}{dt} = \frac{-\left(\frac{\sqrt{2}}{2}\right)10\sqrt{2}\pi}{\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}} + \frac{-\left(\frac{\sqrt{2}}{2}\right)10\sqrt{2}\pi}{\sqrt{25-\left(\frac{\sqrt{2}}{2}\right)^2}}.$$

[9] **11.** Blank papers were common. Many students got the wrong cost function; among those who did, a number wrote it down and stopped. Many tried to equate $12 - x$ with $\sqrt{9 - x^2}$. Those who managed the derivative and critical-point identifications correctly often neglected to justify why the critical point gives a true minimum (the first-derivative test is recommended for this) and/or to report their findings in a user-friendly concluding sentence.

[9] **12.** Almost everyone attempted this question, and most managed the derivatives correctly. A number of writers seemed to overlook the request to find the greatest speed. Other common errors were neglecting the domain endpoints [or the critical points!], or testing the critical point $t = 7/2$ that lies outside the domain.

[9] **13.** This question was dropped from all grade calculations because the version presented to students used the notation log instead of ln. Many students assumed that $\log \equiv \ln$ in calculus (as the examiners intended—the only interpretation consistent with the given expression for y''), but many more interpreted the function as $\log_{10}$. Even though the functions ln and $\log_{10}$ are constant multiples of each other, there is a significant difference between the graphs based on the two interpretations. During the process of carefully marking all papers against a rubric that honoured either choice (provided it was used consistently), it became clear that this ambiguity had prevented some students from showing what they really knew. Dropping the question seemed the fair thing to do. Many students confronted with $x^2 = 1$ found only the single solution $x = 1$; confusion about the definition of an asymptote was nearly universal. (Many students asserted that if a graph ever touches a line, then that line is disqualified from any possible status as an asymptote. The graph of $y = 1/x^2 - 1/x$ with $x > 0$ and its asymptote $y = 0$ shows that this is not quite right; the curve $y = x^{-1}\sin(x)$ $(x > 0)$ and its asymptote $y = 0$ makes the point even more strikingly.)

[7] **14.** Most of these pages were blank.

(a) Many students guessed. There were few correct pictures.

(b) Many who made progress or correctly found the exact area did not write down the approximate area $(\frac{1}{2}\pi a^2)$. The concept of percentage error generated much confusion; many calculated the ratio of the answers (approximately 117.8).