This examination has 15 pages including this cover.

UBC-SFU-UVic-UNBC Calculus Examination

4 June 2009, 12:00-15:00 PDT

Name: ______________________ **Signature:** ______________________

School: ______________________ **Candidate Number:** ______________________

Rules and Instructions

1. *Show all your work!* Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. Some basic formulas appear on page 2. No other notes, books, or aids are allowed. In particular, *all calculator memories must be empty when the exam begins.*
5. If you need more space to solve a problem on page n, work on the back of page $n - 1$.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

1		6
2		4
3		5
4		7
5		6
6		6
7		8
8		9
9		6
10		9
11		9
12		9
13		9
14		7
Total		100

UBC-SFU-UVic-UNBC Calculus Examination

Formula Sheet for 4 June 2009

Exact Values of Trigonometric Functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Trigonometric Definitions and Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Continued on page 3

[6] **1.** Find the derivative of each function below. Do not simplify.

(a) $f(x) = \dfrac{\sin(5x)}{1+x^2}$

(b) $g(x) = \ln\left(e^{x^2} + \sqrt{1+x^4}\right)$

[4] **2.** Find an equation for the line that is tangent to this curve at the point where $x = 1$:

$$y = \ln\left(\frac{2x-1}{2x+1}\right).$$

[5] **3.** Find an equation for the line tangent to this curve at the point $(2, 1)$:

$$x^2y^3 + x^3 - y^2 = 11.$$

[7] **4.** Let $f(x) = \dfrac{4}{\pi}\arctan(2x)$, and define $\qquad \alpha = f\left(\dfrac{1}{2} + \dfrac{\pi}{10^{10}}\right).$

Clearly, $\alpha \approx f(\frac{1}{2})$ and $f(\frac{1}{2}) = 1$. (Alternative notation for arctan is $\tan^{-1}$.)

(a) Find a more accurate approximation for α.

(b) Decide if your improved approximation is larger or smaller than the exact value of α. Explain.

[6] **5.** Find each limit below or show that it does not exist. Justify your results with algebra, not with your calculator!

(a) $\displaystyle\lim_{x\to 0}\left(\frac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x}\right)$

(b) $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+cx} - x\right)$, where c is a constant. (Answer in terms of c.)

[6] **6.** Identify the largest number in the sequence

$$a_1 = 1,\ a_2 = 2^{1/2},\ a_3 = 3^{1/3},\ a_4 = 4^{1/4},\ \ldots,\ a_n = n^{1/n},\ \ldots.$$

Hint: Calculating each number in this infinite list would take forever. Analyzing the function $f(x) = x^{1/x}$ on a suitable domain is one way to reduce the work required.

[8] **7.** A particle moves along the x-axis, where position is measured in metres. At time $t \geq 0$, measured in seconds, the particle's acceleration is

$$a = 2t - 3.$$

Continued on page 4

At time $t = 0$, the particle has position $x = 10$ and velocity $v = -4$.

(a) At some time $t > 0$, the particle's direction of motion changes. Find the particle's position at this instant.

(b) Find the total distance travelled by the particle during the first 6 seconds of its motion.

[9] **8.** Newton's Best Coffee (NBC) serves a brew that's too hot to drink immediately. Twenty (20) minutes after a cup is served, its temperature is 70°C; waiting another two (2) minutes lets the temperature drop to 68°C. A visitor suggests that since the temperature has dropped two degrees in two minutes, the coffee must have been 90°C when it was served.

(a) Explain in words, with no equations or calculations, why this reasoning is not perfectly accurate.

(b) Decide whether the true serving temperature was higher or lower than 90°C. Explain your decision in words, with no equations or calculations.

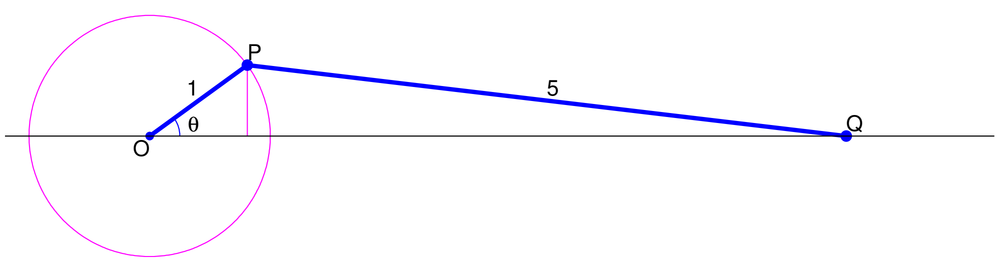
(c) Assuming the room temperature at NBC is 20°C, calculate the coffee's actual serving temperature.

[6] **9.** Consider the function $f(x) = x\sin(|x|)$.

(a) Does $f'(0)$ exist? If so, explain why and calculate it; if not, explain why not.

(b) Does $f''(0)$ exist? If so, explain why and calculate it; if not, explain why not.

[9] **10.** A wheel of radius 1 metre spins counterclockwise around the origin at a constant speed of 10 revolutions per second. One end of a rod 5 metres long pivots on a point P on the wheel's perimeter; the rod's other end, Q, slides back and forth along the x-axis. See the sketch. Find the linear speed of point Q at the instant when the angle θ shown in the sketch is $\pi/4$ radians.

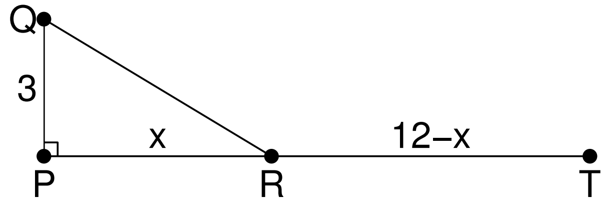


(*Hint*: Split $\left|\overline{OQ}\right| = u + w$, where u and w are formed by dropping a perpendicular from P to $\overline{OQ}$. You can find u and w from basic trigonometry.)

[9] **11.** Residents of island Q need a new fibre-optic cable for their network. The island is 3 km offshore from the nearest point P on a straight coastline, and the nearest broadband signal source is in town T, 12 km along the shore from P. See the sketch. Underwater cable costs twice as much as dry-land cable, so the islanders decide to save money

Continued on page 5

by running underwater cable from Q to R and dry-land cable from R to T. What location for point R gives the lowest cost?



[9] **12.** A particle moves along a vertical line, starting at time $t = 0$ and finishing at time $t = 3$. Its height at time t is

$$y = 4t^3 - 24t^2 + 21t, \qquad 0 \le t \le 3.$$

Find the highest and lowest points reached by this particle, and find when its speed is greatest. Give full reasons for your conclusions.

[9] **13.** Using the axes provided on the next page, make a reasonable sketch of the curve

$$y = 4x + 2 - 5\ln(1 + x^2). \qquad (\textit{Hint}:\ (1 + x^2)^2 y'' = 10(x^2 - 1).)$$

Support your sketch with calculations that identify the following features:

(a) The exact (x, y) coordinates of each critical point.

(b) Exact intervals on which the curve is increasing or decreasing.

(c) The exact (x, y) coordinates of each inflection point.

(d) Is it correct to say, “The line $y = 4x + 2$ is a slant asymptote for this curve?” Why or why not?

[7] **14.** Eddie is in a hurry to find the area A that lies above the x-axis and below the curve

$$y = \frac{(a^2 - x^2)}{a}.$$

(Here $a > 0$ is a constant.) Observing that the curve is concave down and passes through three points that also lie on the semicircle $y = \sqrt{a^2 - x^2}$, Eddie decides to approximate A using the area between the x-axis and the semicircle.

(a) Is Eddie’s approximation larger or smaller than the true value of A?

(b) Find the exact value of A and use it to calculate the percentage error in Eddie’s approximation.

The End