SFU – UBC – UNBC – Uvic
Calculus Challenge Examination
June 5, 2008, 12:00 – 15:00

Host: SIMON FRASER UNIVERSITY

First Name: ____________________________

Last Name: ____________________________

School: ____________________________

Student signature

INSTRUCTIONS

1. Show all your work. Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete.
2. Calculators are optional, not required. Correct answer that is calculator ready, like $3+\ln 7$ or e^2, are preferred.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
5. If you need more space to solve a problem on page *n*, work on the back of the page *n*-1.
6. CAUTION – Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.

Question	Maximum	Score
1	9	
2	6	
3	6	
4	6	
5	8	
6	5	
7	6	
8	6	
9	6	
10	8	
11	8	
12	9	
13	9	
14	8	
Total	100	

[9] 1. In each case either compute the limit explaining briefly how you obtained the value, or explain why the limit does not exist.

(a) $\lim_{x\to 0}\frac{3}{4^{1/x}+1}$

$\lim_{x\to 0^+}\frac{3}{4^{1/x}+1}\left(=\frac{3}{4^{+\infty}+1}=\frac{3}{\infty}\right)=0$ and $\lim_{x\to 0^-}\frac{3}{4^{1/x}+1}\left(=\frac{3}{4^{-\infty}+1}=\frac{3}{0+1}\right)=3$ so $\lim_{x\to 0}\frac{3}{4^{1/x}+1}$ DNE.

(b) $\lim_{x\to -\infty}\left(\frac{x^4-5}{x^3+2x^2}-\frac{x^5+1}{x^4-1}\right)$

$$\begin{aligned}\lim_{x\to -\infty}\left(\frac{x^4-5}{x^3+2x^2}-\frac{x^5+1}{x^4-1}\right) &= \lim_{x\to -\infty}\left(\frac{\left(x^4-5\right)\left(x^4-1\right)-\left(x^5+1\right)\left(x^3+2x^2\right)}{\left(x^3+2x^2\right)\left(x^4-1\right)}\right)\\ &= \lim_{x\to -\infty}\left(\frac{\left(x^8-6x^4+5\right)-\left(x^8+2x^7+x^3+2x^2\right)}{\left(x^3+2x^2\right)\left(x^4-1\right)}\right)\\ &= \lim_{x\to -\infty}\left(\frac{-2x^7-6x^4-x^3-2x^2+5}{\left(x^3+2x^2\right)\left(x^4-1\right)}\right)\\ &= \frac{-2}{1}=-2\end{aligned}$$

since the degree of the numerator is the same as the degree of the denominator.

(c) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}=\lim_{x\to\infty}\left(1+\frac{1}{x/3}\right)^{6(x/3)}=\left(\lim_{x\to\infty}\left(1+\frac{1}{x/3}\right)^{x/3}\right)^6=e^6$$

[6] 2. Find the derivatives of the functions below. Do not simplify.

(a) $g(x)=\sec\left(5x^2-\tan(2x)\right)$

$g'(x)=\sec\left(5x^2-\tan(2x)\right)\tan\left(5x^2-\tan(2x)\right)\left[10x-2\sec^2(2x)\right]$

(b) $h(x)=5\left(\sqrt[3]{x}+1\right)e^{x^2}$

$h'(x)=\frac{5}{3}x^{-2/3}e^{x^2}+10x\left(\sqrt[3]{x}+1\right)e^{x^2}$

[6] 3. Given $x^{\cos y}=y^{\sin x}$ use logarithmic differentiation to find an expression for $\frac{dy}{dx}$ in terms of x and y. No need to simplify the expression.

$$\ln x^{\cos y} = \ln y^{\sin x}$$
$$\cos y \ln x = \sin x \ln y$$
$$\frac{d}{dx}\cos y \ln x = \frac{d}{dx}\sin x \ln y$$
$$-\sin y \cdot \frac{dy}{dx} \cdot \ln x + \cos y \cdot \frac{1}{x} = \cos x \ln y + \sin x \cdot \frac{1}{y} \cdot \frac{dy}{dx}$$
$$\left(\frac{\sin x}{y} + \sin y \ln x\right)\frac{dy}{dx} = \frac{\cos y}{x} - \cos x \ln y$$
$$\frac{dy}{dx} = \left(\frac{\cos y}{x} - \cos x \ln y\right) \div \left(\frac{\sin x}{y} + \sin y \ln x\right)$$

[6] 4. The limit $\lim_{h\to 0}\frac{\sqrt{9-2h}-3}{h}$ represents the derivative of some function *f* at the point $x=0$.

(a) Find one possible function definition for *f* such that $f'(0) = \lim_{h\to 0}\frac{\sqrt{9-2h}-3}{h}$.

$f'(0) = \lim_{h\to 0}\frac{\sqrt{9-2h}-3}{h} = \lim_{h\to 0}\frac{\sqrt{9-2(0+h)}-\sqrt{9-2(0)}}{h}$ so in general $f(x) = \sqrt{9-2x}+c$ for any real number *c*.

(b) Evaluate the limit directly without using the fact that it is equal to $f'(0)$.

$$\lim_{h\to 0}\frac{\sqrt{9-2h}-3}{h} = \lim_{h\to 0}\frac{\sqrt{9-2h}-3}{h}\cdot\frac{\sqrt{9-2h}+3}{\sqrt{9-2h}+3} = \lim_{h\to 0}\frac{9-2h-9}{h\left(\sqrt{9-2h}+3\right)}$$
$$= \lim_{h\to 0}\frac{-2h}{h\left(\sqrt{9-2h}+3\right)} = \lim_{h\to 0}\frac{-2}{\sqrt{9-2h}+3} = \frac{-2}{\sqrt{9}+3} = \frac{-2}{6} = \frac{-1}{3}$$

[8] 5. What is an equation for the straight line through the point $(3,0)$ that is tangent to the graph of $y = x + \frac{3}{x}$ at a point in the first quadrant?

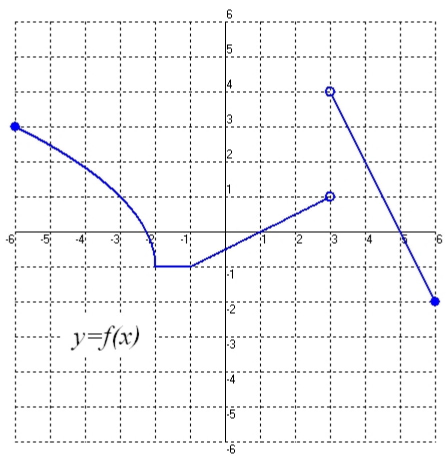
Let $\left(a, a+\frac{3}{a}\right)$ be the point of tangency on the graph of $y = x+\frac{3}{x}$. Then the slope *m* can be calculated in two ways: (1) $y' = 1-\frac{3}{x^2}$ and so $m = 1-\frac{3}{a^2} = \frac{a^2-3}{a^2}$. (2) Using the two points $(3,0)$ and $\left(a, a+\frac{3}{a}\right)$ the slope formula yields $m = \frac{a+\frac{3}{a}}{a-3} = \frac{a^2+3}{a(a-3)}$. Setting the two equations equal to each other we get $\frac{a^2-3}{a^2} = \frac{a^2+3}{a(a-3)}$. Solving this equation for *a* we obtain $(a+3)(a-1)=0$. Since *a* must be in the first quadrant $a=1$. Therefore, the tangent line equation is given by $y-0=-2(x-3)$ or $y=-2x+6$.

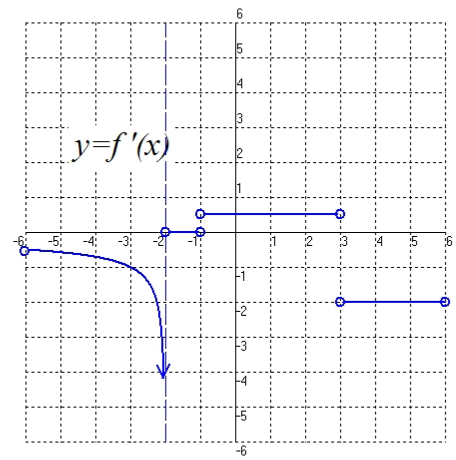
[5] 6. Recall the definition of the inverse tangent function: $\theta = \tan^{-1} t \iff t = \tan\theta$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Show that $\frac{d}{dt}\left(\tan^{-1} t\right) = \frac{1}{1+t^2}$.

$\frac{d}{dt}(t) = \frac{d}{dt}(\tan\theta) \Rightarrow 1 = \sec^2\theta \frac{d\theta}{dt}$. Solving for $\frac{d\theta}{dt}$ we obtain $\frac{d\theta}{dt} = \frac{1}{\sec^2\theta}$, which is defined for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Now, $\frac{1}{\sec^2\theta} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+t^2}$, so $\frac{d\theta}{dt} = \frac{d}{dt}\left(\tan^{-1} t\right) = \frac{1}{1+t^2}$.

[6] 7. Consider the graph of the function $y = f(x)$ shown below to the left. Estimate the slope of the graph of *f* at various points and use these estimates to sketch the graph of $y = f'(x)$.





$f'(-5) \approx -0.5$, $f'(-5) \approx -0.5$, $f'(-3) \approx -1$, $f'(x) = 0$ for $x \in (-2,-1)$, $f'(x) = 0.5$ for $x \in (-1,3)$, $f'(x) = -2$ for $x \in (3,6)$, so the graph is shown above to the right.

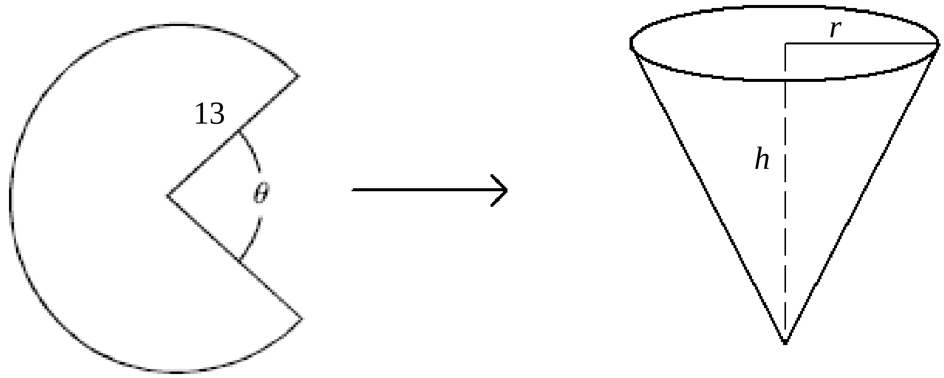
[6] 8. Use linear approximation (or differentials) to estimate $(1.99)^4$.

Using linear approximation $L(x) = f(a) + f'(a)(x-a)$, we choose $f(x) = x^4$, which is differentiable everywhere, and $a = 2$. Then $f(2) = 2^4 = 16$, $f'(x) = 4x^3$, and $f'(2) = 4(2)^3 = 32$, and so the linearization of *f* at *a* is given by $L(x) = 16 + 32(x-2)$. Finally, $f(1.99) = (1.99)^4 \approx L(1.99) = 16 + 32(1.99 - 2) = 15.68$.

[6] 9. Find the largest interval on which the graph of the function $f(x) = \frac{\ln x}{x}$ is concave up.

The domain of *f* is $D_f = (0,\infty)$. $f'(x) = \frac{1-\ln x}{x^2}$ and $f''(x) = \frac{-3+2\ln x}{x^3}$. $f''(x) = 0 \Rightarrow -3 + 2\ln x = 0 \Rightarrow \ln x = 3/2 \Rightarrow x = e^{3/2}$. Now, on the interval $\left(0, e^{3/2}\right)$ we have $f''(x) < 0$, and so by the concavity test *f* is concave down on $\left(0, e^{3/2}\right)$. However, on the interval $\left(e^{3/2}, \infty\right)$ we have $f''(x) > 0$, and so by the concavity test *f* is concave up on $\left(e^{3/2}, \infty\right)$.

[8] 10. A cone is to be constructed from a circular piece of paper with radius 13 centimetres by cutting out a wedge as shown in the diagram below. What is the maximum volume of the cone? (Hint: $V_{cone} = \frac{1}{3}\pi r^2 h$)



We need to maximize $V = \frac{1}{3}\pi r^2 h$. The right diagram shows us that *r* and *h* are related through Pythagoras Theorem $r^2 = 13^2 - h^2 = 169 - h^2$. Then the volume can be expressed in only one variable, namely *h*, as $V = \frac{1}{3}\pi\left(169 - h^2\right)h = \frac{169}{3}\pi h - \frac{1}{3}\pi h^3$ with $h \in [0,13]$. Differentiating the equation with respect to *h* we get $V' = \frac{169}{3}\pi - \pi h^2$. Solving $V' = 0$ for *h* we obtain the two critical numbers $h = \pm\frac{13}{\sqrt{3}}$. However, the negative value must be excluded as only $h = \frac{13}{\sqrt{3}} \in [0,13]$. The maximum volume must occur at either the critical number or an endpoint of the interval. We compare $V(0) = 0$, $V\left(\frac{13}{\sqrt{3}}\right) = \frac{1}{3}\pi\left(169 - \left(\frac{13}{\sqrt{3}}\right)^2\right)\left(\frac{13}{\sqrt{3}}\right) = \frac{4394}{9\sqrt{3}}\pi$, and $V(13) = 0$, and conclude that the maximum volume is $\frac{4394}{9\sqrt{3}}\pi \approx 885.5371562$ cubic units.

[8] 11. A particle is moving along the curve $f(x) = x^2$. As the particle passes through the point $(3, f(3))$, its *x*-coordinate increases at a rate of 5 cm/s. How fast is the distance from the particle to the point $(0, f(0))$ changing at this instant?

Let $y = x^2$ and let *s* denote the distance from the point (x, y) to the point $(0, f(0)) = (0,0)$. Then, $s^2 = x^2 + y^2$. Using the fact that $y = x^2$ this equation simplifies to $s^2 = x^2 + \left(x^2\right)^2 = x^2 + x^4$. We differentiate this equation with respect to time *t*, and obtain $s\frac{ds}{dt} = x\frac{dx}{dt} + 2x^3\frac{dx}{dt} = \left(x + 2x^3\right)\frac{dx}{dt}$. We are given $\left.\frac{dx}{dt}\right|_{(3,9)} = 5$ cm/s. From $s^2 = x^2 + x^4$ and the fact that distance is positive we calculate $s = \sqrt{3^2 + 3^4} = \sqrt{90} = 3\sqrt{10}$. Substituting all these quantities back into the equation $s\frac{ds}{dt}=\left(x+2x^3\right)\frac{dx}{dt}$ we get $3\sqrt{10}\frac{ds}{dt}=\left(3+2(3)^3\right)5$, and so $\frac{ds}{dt}=\frac{95}{\sqrt{10}}$ cm/s.

[9] 12. Sketch the graph of a function *f* with the following properties.

(a) *f* is continuous on its domain $\{x\in\mathbb{R}/x\neq -3,1\}$.

(b) $f(0)=2$ and $f(4)=1$ are inflection points.

(c) $f(3)=4$, $f'(3)=0$, and $f''(3)<0$.

This means that the graph of *f* has a relative maximum point $(3,4)$ by the second derivative test.

(d) $\lim_{x\to\infty} f(x)=-2$ and $\lim_{x\to-\infty} f(x)=-2$.

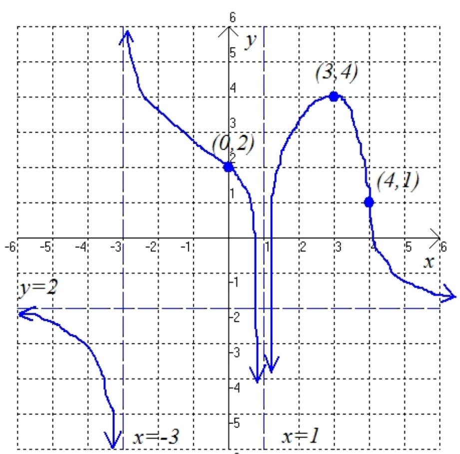
This means that the graph of *f* has as horizontal asymptote $y=-2$.

(e) $\lim_{x\to-3^+} f(x)=\infty$, $\lim_{x\to-3^-} f(x)=-\infty$, $\lim_{x\to1^+} f(x)=-\infty$ and $\lim_{x\to1^-} f(x)=-\infty$.

This means that the graph of *f* has vertical asymptotes $x=-3$ and $x=1$.

(f) $f'(x)<0$ for $x<-3$, $-3<x<1$, and $x>4$ (should be $x>3$), and $f'(x)>0$ for $1<x<4$ (should be $1<x<3$).

This means that the graph of *f* is decreasing for $x<-3$, $-3<x<1$, and $x>3$, and increasing for $1<x<3$.



[9] 13. The levels of a sedative in a patient's blood were monitored to determine the appropriate time for an operation. Every fifteen minutes a blood sample was taken to determine the concentration C of the sedative in milligrams per litre, and then recorded in the table of data shown below.

Time (min)	Concentration C (mg/l)
0	20
15	10.21
30	5.15
45	2.68
60	1.31
75	0.72

(a) Estimate the rate of change of concentration with respect to time at 30 minutes and 60 minutes. Is the rate of change of concentration with respect to time t a constant?

There is a variety of way to estimate the rates asked for.

$$\left.\frac{dC}{dt}\right|_{t=30} \approx \frac{C(45)-C(15)}{30} = \frac{2.68-10.21}{30} = -0.251$$

$$\left.\frac{dC}{dt}\right|_{t=60} \approx \frac{C(75)-C(45)}{30} = \frac{0.72-2.68}{30} = -0.065\overline{3}$$

Since -0.251 is not close to $-0.065\overline{3}$ the rate of change of concentration with respect to time is not constant.

(b) Show that the rate of change is roughly proportional to the concentration. Write this relationship as a differential equation leaving the constant of proportionality, k, undetermined.

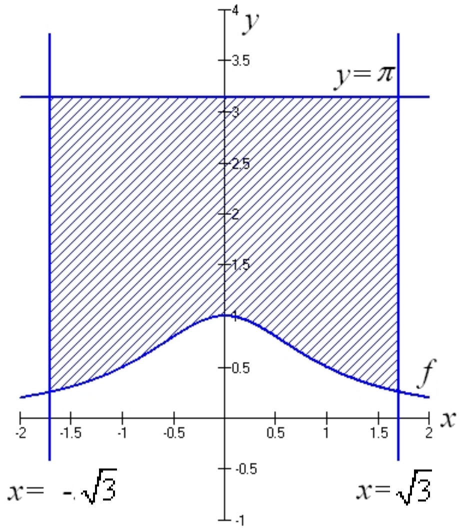
$-0.251 \div C(30) = -0.251 \div 5.15 \approx -0.0487$ and $-0.065\overline{3} \div C(60) = -0.065\overline{3} \div 1.31 \approx -0.0499$ both yield quantities close to each other, which indicates that the rate of change is roughly proportional to the concentration. Therefore, $\frac{dC}{dt} = kC$ for some constant k.

(c) Solve the differential equation from part (b) and choose the constant of proportionality, k, so that the solution satisfies both the entries $C(0) = 20$ and $C(60) = 1.31$ from the table. Write the constant of proportionality accurate to 4 decimal places.

The solution to the initial value problem $\frac{dC}{dt} = kC$, $C(0) = 20$ is $C(t) = 20e^{kt}$. To obtain the constant of proportionality k, we use the given data $C(60) = 1.31$ and solve $1.31 = 20e^{60k}$ for k:

$$1.31 = 20e^{60k} \Leftrightarrow k = \frac{\ln(1.31/20)}{60} \approx -0.0454 \text{. Therefore, } C(t) = 20e^{-0.454t} \text{.}$$

[8] 14. Below is the graph of $f(x)=\dfrac{1}{1+x^2}$.



(a) Graph and shade the region enclosed by the curves $x=\pm\sqrt{3}$, $y=\pi$, and $f(x)=\dfrac{1}{1+x^2}$.

(b) Find the area of the region described in part (a). (Hint: You may use information from a previous question on this exam.)

Using symmetry, the area of the shaded region can be calculated with the following integral to be $2\int_0^{\sqrt{3}} \pi-\dfrac{1}{1+x^2}\,dx = 2\left[\pi x-\tan^{-1}x\right]_0^{\sqrt{3}} = 2\left[\left(\pi\sqrt{3}-\tan^{-1}\sqrt{3}\right)-(0)\right] = 2\left(\sqrt{3}\pi-\dfrac{\pi}{3}\right)$ or $\dfrac{2\pi}{3}\left(3\sqrt{3}-1\right)$ square units.