SFU - UBC – UNBC - UVic

Calculus Challenge Exam

June 8, 2006, 12 noon to 3 p.m.

Host: SIMON FRASER UNIVERSITY

Student signature

INSTRUCTIONS

1. Show all your work. Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete.
2. Calculators are optional, not required. Correct answer that is calculator ready, like $3+\ln 7$ or e^2, are preferred.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
5. If you need more space to solve a problem on page *n*, work on the back of the page *n*-1.
6. CAUTION – Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:

   (a) Using any books, papers or memoranda.

   (b) Speaking or communicating with other candidates.

   (c) Exposing written papers to the view of other candidates.

Question	Maximum	Score
1	4	
2	10	
3	6	
4	9	
5	6	
6	15	
7	4	
8	6	
9	4	
10	8	
11	8	
12	6	
13	4	
14	4	
15	6	
Total	100	

[2] 1. (a) Sketch a rough graph of $y(x) = \dfrac{|x-2|}{x-2}$ near $x = 2$.

ANSWER:

[2] (b) Evaluate $\lim_{x \to 2^-} \dfrac{|x-2|}{x-2}$.

JUSTIFY YOUR ANSWER

[6] 2. (a) Given that

$$f(x) = \frac{x^2 e^{4x}}{\sin x + \cos(2-3x)}$$

find $f'(x)$. No simplification is necessary.

ANSWER

[4] (b) Suppose that functions $F(x)$ and $G(x)$ satisfy the following properties:

$$F(3) = 2, \quad G(3) = 4, \quad G(0) = 3,$$
$$F'(3) = -1, \quad G'(3) = 0, \quad G'(0) = 2.$$

If $T(x) = F(G(x))$ and $U(x) = \ln(F(x))$, find $T'(0)$ and $U'(3)$.

ANSWER:

SHOW YOUR WORK

[3] 3. (a) Let $f(x) = xe^x$. Express $f'(0)$ as a limit by using the definition of derivative and hence evaluate $f'(0)$.

ANSWER:

SHOW YOUR WORK

ANSWER:

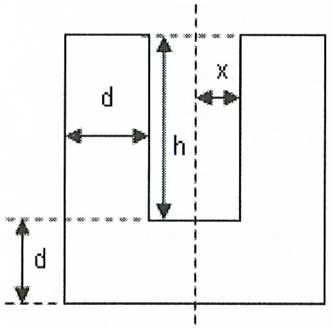
[3] (b) Let $g(x) = \left|1 - x^2\right|$. Does $g'(x)$ exist at $x = 1$?

EXPLANATION

[9] 4. An open steel cylinder is required which will hold V (cm^3) of liquid. The thickness of the walls and base of the cylinder is d (cm). Find the height h (cm) and inner radius r (cm) of the cylinder which minimize the amount of steel required. The diagram shows the cross-section of the cylinder through its axis. HINT: Express M, the volume of steel required, as a function of x.

ANSWER:

SHOW YOUR WORK



5. A car moves in a straight line. At time t (measured in seconds) its position (measured in metres) is

$$s(t) = \frac{1}{100}t^3, \quad 0 \le t \le 5.$$

ANSWER:

[2] (a) Find its average velocity between $t = 0$ and $t = 5$.

JUSTIFY YOUR ANSWER

ANSWER:

[2] (b) Find its instantaneous velocity for $0 < t < 5$.

JUSTIFY YOUR ANSWER

ANSWER:

[2] (c) At what time is the instantaneous velocity of the car equal to its average velocity?

JUSTIFY YOUR ANSWER

6. Consider the function $f(x) = x^2 e^{1/x}$.

[8] (a) Find the following (no explanation is required in parts i) - iv)):

i. $\lim_{x \to 0^-} x^2 e^{1/x}$

ANSWER:

ii. $\lim_{x \to 0^+} x^2 e^{1/x}$

ANSWER:

iii. $\lim_{x \to \infty} x^2 e^{1/x}$

ANSWER:

iv. $\lim_{x \to -\infty} x^2 e^{1/x}$

ANSWER:

v. $f'(x)$

ANSWER:

SHOW YOUR WORK

vi. $f''(x)$

ANSWER:

SHOW YOUR WORK

[5] b) Using your findings, find the following

i. The $x-$ and $y-$ Intercepts (if any)

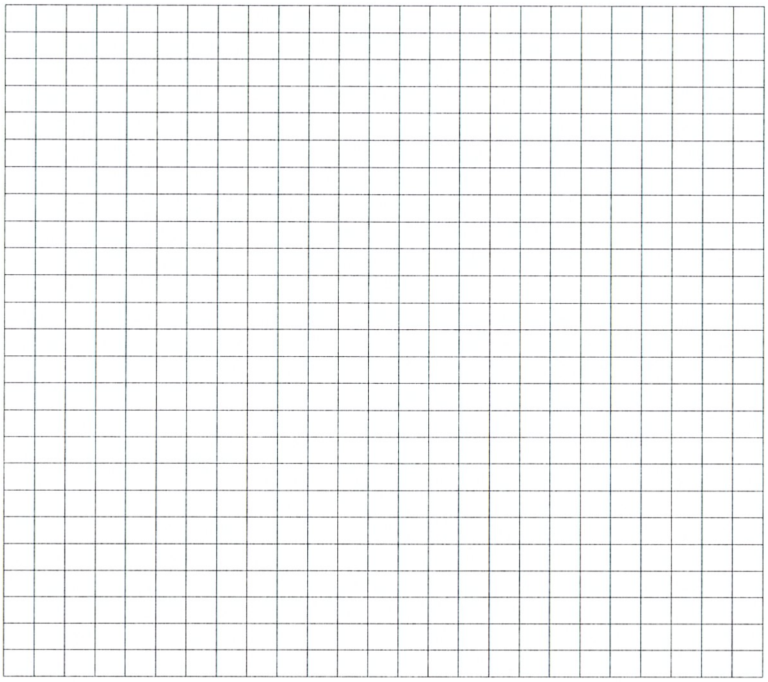
ii. The local maxima and minima (if any)

iii. The Inflection points (if any)

iv. The largest intervals on which the graph is concave up and concave down

v. The asymptotes

[2] (c) Sketch a rough graph of $f(x)$ indicating your findings:



[4] 7. A copper cube of side 5 *cm* is shaved on all sides to produce a cube of side $(5-\varepsilon)$ *cm*. Given that the density of copper is 8.96 g/cm^3 and that the shaving decreases the mass of the cube by 0.96 *g*, use a linear approximation to estimate ε.

ANSWER:

JUSTIFY YOUR ANSWER

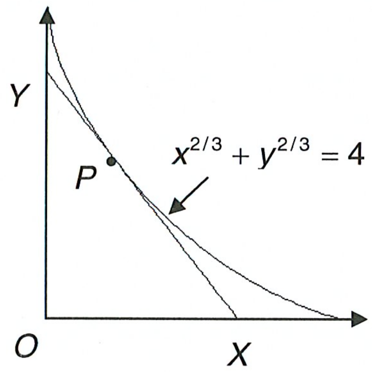
[6] 8. Consider the curve

$$x^{2/3} + y^{2/3} = 4$$

in the first quadrant. Show that the length of a segment XY of a tangent line to the curve at a point P cut off by the coordinate axes is constant and find this length.

ANSWER:

EXPLANATION



[4] 9. Using logarithmic differentiation, find $y'(0)$ given that

$$y(x)=\frac{\sqrt{1+2x}\,\sqrt[4]{1+4x}}{\sqrt[3]{1+3x}\,\sqrt[5]{1+5x}\,\sqrt[7]{1+7x}}$$

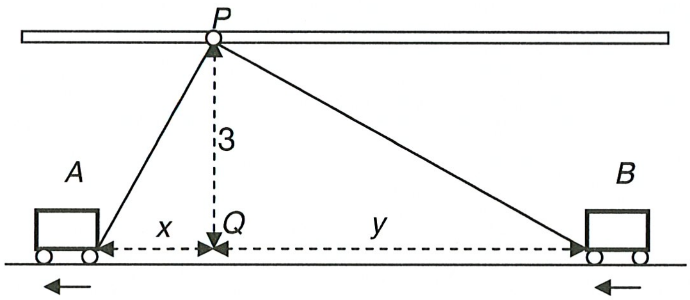
ANSWER:

SHOW YOUR WORK

[8] 10. Two carts, *A* and *B*, are on the floor of a warehouse. The carts are connected by a rope 25 metres long. The rope is stretched tight and pulled over a pulley attached to a rafter 3 metres above a point *Q* between the carts (see picture). How fast is the distance y between the cart *B* and point *Q* changing when cart *A* is 4 metres away from *Q* and is being pulled away from *Q* at a speed of 2 metres per second?

ANSWER:

JUSTIFY YOUR ANSWER



[5] 11. (a) Find the general solution of the differential equation

$$\frac{dy}{dx} = y(y-2).$$

ANSWER:

SHOW YOUR WORK

[2] (b) Find the particular solution of the above equation that satisfies initial condition $y(0) = 0.5$.

ANSWER:

SHOW YOUR WORK

[1] (c) What happens to the value of $y(x)$ found in Part (b) as $x \to \infty$?

EXPLAIN YOUR ANSWER

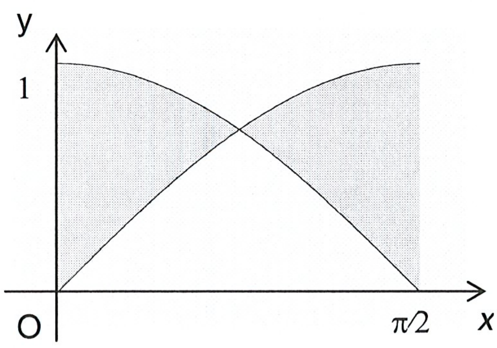
[6] 12. Find area of the shaded region included between the curves

$$y = \sin x \text{ and } y = \cos x$$

between $x = 0$ and $x = \pi/2$.

ANSWER:

SHOW YOUR WORK



[4] 13. Given that $f''(t) = e^{-2t} + \cos\frac{1}{2}t$ for all t, find an explicit formula for $f(t)$ if $f(0) = f'(0) = 0$.

ANSWER:

SHOW YOUR WORK

[4] 14. Find

$$\lim_{x \to 0} \frac{\sec x - 1}{x \sec x}.$$

ANSWER:

SHOW YOUR WORK

[6] 15. Use the chain rule to show that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

EXPLANATIONS