[5] **1.** Find the tangent line to the curve $y = \dfrac{\sqrt{x}}{1 + x^2}$ at the point where $x = 1$.

Solution:

$$\text{Quotient Rule:} \qquad y'(x) = \frac{\frac{1}{2\sqrt{x}}\left(1 + x^2\right) - \sqrt{x}(2x)}{(1 + x^2)^2}$$

$$\text{Evaluation:} \qquad m = y'(1) = \frac{\frac{1}{2}(2) - 2}{2^2} = -\frac{1}{4}$$

$$\text{Line:} \qquad y = \frac{1}{2} - \frac{1}{4}(x - 1) = -\frac{1}{4}x + \frac{3}{4}$$

Discussion: Many students made a sign error in quoting the quotient rule.

[5] **2.** Given $f(x) = \cos\left(e^{x^2} + \sin(\ln(x))\right)$, find $f'(x)$.

[The correct answer will receive full credit even if no justification is shown.]

Solution: $\qquad f'(x) = -\sin\left(e^{x^2} + \sin\left(\ln(x)\right)\right)\left[2xe^{x^2} + \dfrac{1}{x}\cos(\ln(x))\right]$

Discussion:

- Omitting the outside brackets in $[2xe^{x^2} + 1/x\cos(\ln(x))]$ messes up the order of operations rules and changes the value of the result.

- Likewise when applying the chain rule: if $1/x$ ends up *outside* the brackets, multiplying the whole expression, the meaning is spoiled.

- There were a number of classic function-notation mistakes: $\sin(a + b) = \sin(a) + \sin(b)$, etc.

[5]   **3.**   Find the exact value of $L = \lim\limits_{h \to 0} \dfrac{(e^h - 1)\sin(3h)}{h^2}$.

Solution: Students are expected to recognize the definition of derivative.

$$L = \lim_{h \to 0} \frac{e^h - 1}{h} \cdot \lim_{h \to 0} \frac{\sin(3h)}{h} = \frac{d}{dx}\left(e^x\right)\Big|_{x=0} \cdot \frac{d}{dx}\left(\sin(3x)\right)\Big|_{x=0}$$
$$= \left(e^0\right) \cdot (3\cos(0)) = 3.$$

L'Hospital's Rule provides an acceptable alternate method:

$$L \overset{\left[\frac{0}{0}\right]}{=} \lim_{h \to 0} \frac{e^h \sin(3h) + 3(e^h - 1)\cos(3h)}{2h}$$
$$\overset{\left[\frac{0}{0}\right]}{=} \lim_{h \to 0} \frac{\left[e^h \sin(3h) + 3e^h \cos(3h)\right] + \left[3e^h \cos(3h) - 9(e^h - 1)\sin(3h)\right]}{2}$$
$$= \frac{0 + 3 + 3 + 0}{2} = 3.$$

Discussion:

[5]   **4.**   Find all points where this curve intersects its horizontal asymptote:

$$y = \frac{x^3 - 2x^2 + 5}{x^2 - x^3}.$$

Solution:

Horizontal Asymptote: $y = \lim\limits_{x \to \infty} \dfrac{x^3 - 2x^2 + 5}{x^2 - x^3} \cdot \dfrac{1/x^3}{1/x^3} = \lim\limits_{x \to \infty} \dfrac{1 - 2/x + 5/x^3}{1/x - 1} = -1.$

Intersection: $-1 = \dfrac{x^3 - 2x^2 + 5}{x^2 - x^3} \iff x^3 - x^2 = x^3 - 2x^2 + 5$

$\iff x^2 = 5 \iff x = \pm\sqrt{5}.$

Points: $(-\sqrt{5}, -1), \ (\sqrt{5}, -1)$ [recall $y = -1$ from above].

Discussion: Some writers tried to find the *vertical* asymptotes for the given curve. No points were given for this, even when they succeeded.

[8] **5.** Consider this curve C: $y^4 - 4y^2 = x^4 - 9x^2$.

(a) Show that the point $(3, 2)$ lies on C.

(b) Use a suitable tangent line to estimate a value of b for which $(3.02, b)$ lies on C.

Solution: (a) At $(3, 2)$,

$$\text{LHS} = \left[y^4 - 4y^2 \right]_{y=2} = 16 - 16 = 0 \qquad \text{and} \qquad \text{RHS} = \left[x^4 - 9x^2 \right]_{x=3} = 81 - 81 = 0.$$

Since the equation defining C is satisfied by $(3, 2)$, this point lies on C.

(b) Treating y as a function of x, differentiate:

$$4y^3 y' - 8yy' = 4x^3 - 18x.$$

Plug in $(3, 2)$, re-interpreting y' as $\left. \dfrac{dy}{dx} \right|_{(3,2)}$ from now on:

$$[32 - 16]y' = 108 - 54 \qquad \Longleftrightarrow \qquad y' = \frac{54}{16} = \frac{27}{8}.$$

The tangent line to C at $(3, 2)$ is

$$y = 2 + \frac{27}{8}(x - 3).$$

Using $x = 3.02$ in the tangent line gives the approximation

$$b \approx 2 + \frac{27}{8}\left(\frac{1}{50}\right) = 2 + \frac{27}{400} = \frac{827}{400} = 2.0675.$$

Discussion: The exact value is $b = \dfrac{1}{50}\sqrt{5000 + \sqrt{31863101}}$, which is approximately 2.063467.

(a) Interesting conceptual error: trying to prove that $(3, 2)$ lies on the curve by differentiating the equation, then plugging $(3, 2)$ into LHS and RHS then saying something like "hence we see that y' exists at $(3, 2)$ so $(3, 2)$ must be on the curve."

(b) Omitted y' in the implicit differentiation; Plugging (3.02,2) into the y' expression; Confusing b and the tangent line's intercept; Various algebra mistakes.

[8]   **6.**   Among all circular cylinders with diagonal length equal to 6, find the height of the one whose volume is largest. Explain how you can be sure that your result gives the *maximum* volume.

Background: The diagonal length L of a cylinder is the length of the longest straight rod that will fit inside it. (See the rough sketch at right.)

Hint: Let the desired cylinder have base radius r, and height h. Start by finding a relationship between r and h, based on the fact that $L = 6$.
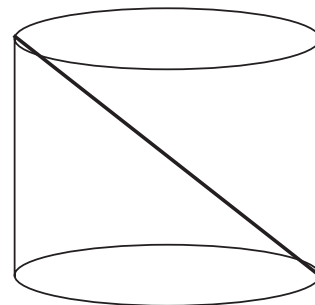
Solution: Write L along the diagonal in the sketch provided, $2r$ for the cylinder's base diameter, and h for its height. By Pythagoras,

$$L^2 = h^2 + (2r)^2.$$

Use $L = 6$, as given:

$$36 = h^2 + 4r^2 \quad \Longrightarrow \quad r^2 = \frac{1}{4}(36 - h^2).$$

Deduce the volume formula and appropriate domain:

$$V(h) = \pi r^2 h = \frac{\pi}{4}(36 - h^2)h, \qquad 0 \le h \le 6.$$

Calculate:

$$V'(h) = \frac{\pi}{4}(36 - 3h^2) = \frac{3\pi}{4}(12 - h^2).$$

Find critical points:

$$V'(h) = 0 \quad \Longleftrightarrow \quad h = \pm\sqrt{12}.$$

Apply first-derivative test:

$$V'(h) > 0 \text{ for } 0 < h < \sqrt{12} \qquad \text{and} \qquad V'(h) < 0 \text{ for } \sqrt{12} < h < 6,$$

so $h = \sqrt{12} = 2\sqrt{3}$ gives the maximum volume.

Discussion: The major problem here was not explaining clearly WHY the max occurs at $h = \sqrt{12}$–either by using first or second derivatives, or by evaluation at endpoints and noting that there is one CP only. Common faulty explanations:

$V'(\sqrt{12}) = 0$ so it must be a max.

appealing to use of a graphing calculator

NOTE: several people also did this question without using calculus at all - i.e. by sampling at few random points.

[8] **7.** A particle travels along the x-axis. When the particle is at the point x, its velocity is

$$\frac{dx}{dt} = \frac{1}{1+2x}.$$

Find the particle's acceleration at the instant when its position is $x = 2$.

Solution:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}(1+2x)^{-1}$$

$$= -(1+2x)^{-2}\left[2\frac{dx}{dt}\right]$$

$$= -\frac{2}{(1+2x)^3}.$$

At the instant when $x = 2$, the particle's acceleration is $a = -\dfrac{2}{(5)^3} = -\dfrac{2}{125}$.

Discussion: The overwhelming majority of writers calculate the derivative of the given expression with respect to x, plugged in $x = 2$, and reported an answer of $-2/25$. These received a (generous) score of 4 marks instead of the 8 they could have earned by recognizing x as a time-varying quantity and applying the chain rule to differentiate with respect to t.

[8]　**8.**　A vegan casserole with a temperature of $20\,^\circ$C is placed in an oven whose constant temperature is $200\,^\circ$C. The casserole's temperature reaches $30\,^\circ$C ten minutes later. How long must the casserole spend in the oven (total cooking time) before it is ready to serve? Base your answer on these assumptions:

(i)　Serving temperature for the casserole is $90\,^\circ$C.

(ii)　The rate of change of the casserole's temperature is proportional to the difference between its temperature and the temperature of the oven.

NOTE: A "calculator-ready" answer is fully acceptable.

Solution: Let $u(t)$ denote the casserole's temperature at time t. Assumption (ii)—Newton's law of cooling—says

$$\frac{du}{dt} = k(200 - u)$$

for some constant k. Let $y(t) = 200 - u(t)$ so

$$y'(t) = 0 - u'(t) = -k(200 - u(t)) = -ky(t), \quad \text{giving} \quad y(t) = Ae^{-kt}.$$

Use the starting temperature to find

$$A = y(0) = 200 - 20 = 180,$$

then use the intermediate temperature to find

$$170 = 200 - 30 = y(10) = 180e^{-10k} \quad \Longrightarrow \quad -10k = \ln\left(\frac{17}{18}\right) \quad \Longrightarrow \quad k = \frac{1}{10}\ln(18/17).$$

Finally, solve for the serving time T using

$$110 = 200 - 90 = y(T) = Ae^{-kT} \quad \Longleftrightarrow \quad -kT = \ln(110/A)$$

$$\Longleftrightarrow \quad T = \frac{1}{k}\ln(A/110) = \frac{1}{10}\frac{\ln(18/11)}{\ln(18/17)}.$$

Equivalent forms:

$$T = \frac{1}{10}\frac{\ln(11/18)}{\ln(17/18)} = \frac{1}{10}\frac{\ln(18) - \ln(11)}{\ln(18) - \ln(17)}.$$

Calculator approximation: $T \approx 86.16$ minutes.

Discussion:

[8] **9.** A particle travels in the plane. At each time t, its (x, y)-coordinates are given by

$$x = t - \sin t, \qquad y = \cos t.$$

How fast is the particle's distance from the origin increasing at the instant when the particle is at the point $(\pi, -1)$?

Solution: The particle is at $(\pi, -1)$ at the instant when both

$$(1) \quad \pi = x = t - \sin(t), \qquad (2) \quad -1 = y = \cos(t).$$

Equation (2) implies that $t = \pi + k2\pi$ for some integer k, which obeys equation (1) exactly when $\pi = \pi + k2\pi$, i.e., $k = 0$. So the instant of interest is $t = \pi$.

If D denotes the particle's distance from the origin, then

$$D^2 = x^2 + y^2 = (t - \sin(t))^2 + (\cos(t))^2 \qquad (*)$$

at all times. Differentiation gives

$$2D\frac{dD}{dt} = 2(t - \sin(t))[1 - \cos(t)] + 2(\cos(t))[-\sin(t)]. \qquad (**)$$

At $t = \pi$, line $(*)$ gives $D^2 = \pi^2 + 1$, so $D = \sqrt{\pi^2 + 1}$. Line $(**)$ (after cancelling 2's) gives

$$\sqrt{\pi^2 + 1}\frac{dD}{dt} = (\pi)[2] + (-1)[0], \quad \text{i.e.,} \quad \frac{dD}{dt} = \frac{2\pi}{\sqrt{\pi^2 + 1}}.$$

The particle's distance from the origin is increasing at $2\pi/\sqrt{\pi^2 + 1}$ units of distance per unit time at the instant in question.

Discussion: Many found an explicit expression for D in terms of t, and differentiated. The expression for D is more complicated than the expression for D^2, so this approach produced more numerical errors.

Quite a few wrote $D^2 = x^2 + y^2$, and concluded that $2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$, and found x, $\frac{dx}{dt}$, and so on at $t = \pi$ directly. This is efficient and produced very few errors.

A number of students calculated $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. This is the particle's *speed*, which is *usually not* the rate at which the particle's distance from the origin is changing. (Concept practice: in what situations do these two quantities agree?)

[8] **10.** Let $f(x) = e^{\left(10x - \frac{x^2}{2}\right)}$.   Find all intervals over which $f(x)$ is concave up.

Solution:

$$f'(x) = e^{\left(10x - \frac{x^2}{2}\right)}[10 - x]$$

$$f''(x) = e^{\left(10x - \frac{x^2}{2}\right)}[10 - x]^2 - e^{\left(10x - \frac{x^2}{2}\right)}$$

$$= e^{\left(10x - \frac{x^2}{2}\right)}\left[(100 - 20x + x^2) - 1\right]$$

$$= e^{\left(10x - \frac{x^2}{2}\right)}\left[x^2 - 20x + 99\right]$$

$$= e^{\left(10x - \frac{x^2}{2}\right)}(x - 11)(x - 9).$$

The exponential function always returns positive values, so

$$f''(x) > 0 \quad \Longleftrightarrow \quad (x - 11)(x - 9) > 0 \quad \Longleftrightarrow \quad x < 9 \text{ or } x > 11.$$

Thus f is concave up on the interval $(-\infty, 9]$ and on the interval $[11, +\infty)$.

Discussion: There were many differentiation errors, especially in the calculation of $f''(x)$.

Whether or not the endpoints 9 and 11 should be included in the intervals in question is a matter of taste, and open intervals were accepted for full credit. However, it is not correct to say that f is concave up on the *set* $S = (-\infty, 9] \cup [11, +\infty)$: concavity is a geometric property (not an algebraic inequality) and its definition is not satisfied on the set S.

[8] **11.** A particle travels along the x-axis. At each time t, the particle's acceleration is

$$a(t) = 4e^{-2t}.$$

At time $t = 0$, the particle's velocity is 3, and its x-coordinate is 7.

(a) Find a formula for the velocity of the particle at time t.

(b) Find the particle's x-coordinate at time $t = 10$.

Solution: (a)
$$\frac{dv}{dt} = a(t) = 4e^{-2t}$$
$$\implies v = \int dv = \int a\, dt = \int 4e^{-2t}\, dt$$
$$\implies v = -2e^{-2t} + C \quad \text{for some constant } C.$$

Since $3 = v(0) = -2e^0 + C$, we have $C = 5$ and $v(t) = 5 - 2e^{-2t}$.

(b)
$$\frac{dx}{dt} = v(t)$$
$$\implies x = \int dx = \int v(t)\, dt = \int \left(5 - 2e^{-2t}\right) dt$$
$$\implies x = 5t + e^{-2t} + K \qquad \text{for some constant } K.$$

Since $7 = x(0) = e^0 + K$, we have $K = 6$ and $x(t) = 6 + 5t + e^{-2t}$. At time $t = 10$, the particle's x-coordinate is $x(10) = 56 + e^{-20}$.

Discussion: The integrations were often done incorrectly. All too many times people believed that $\int e^{-2t}\, dt$ is equal to $-2e^{-2t} + C$. When one integrates, it is useful to quickly differentiate the presumed answer to check for errors.

It should not be necessary to use substitution to evaluate such a simple integral.

Regrettably, a number of people omitted the "C."

Fairly often, it was assumed that e^{-2t} is 0 at $t = 0$. This made for errors in evaluating the constants of integration.

[8] **12.** A small company hires students to gather wild mushrooms. If the company uses L hours of student labour per day, it can harvest $3L^{2/3}$ kilograms (Kg) of wild mushrooms, which it can sell for \$15.00 per Kg. The company's only costs are for labour. It pays its pickers \$6.00 per hour, so L hours of labour cost the company $6L$ dollars.

How many hours L of labour should the company use per day in order to maximize profit? (Note that daily profit is equal to daily revenue minus daily cost.) Show that your answer really does give maximum profit.

Solution:
$$\text{Profit} = \text{Revenue} - \text{Cost}$$
$$= \left(15 \text{ \$/Kg}\right)\left(3\,L^{2/3} \text{ Kg}\right) - \left(6 \text{ \$/hr}\right)\left(L \text{ hr}\right)$$
$$= 45L^{2/3} - 6L \qquad \text{(dollars)}.$$

Let $f(x) = 45x^{2/3} = 6x$ and seek a maximum over $x > 0$:

$$f'(x) = 45\left(\frac{2}{3}\right)x^{-1/3} - 6 = 30x^{-1/3} - 6 = 6x^{-1/3}\left(5 - x^{1/3}\right).$$

There is just one critical point ($f'(x) = 0$), at $x^{1/3} = 5$ or $x = 125$, and the factored form of f' above makes it easy to deduce

$$f'(x) > 0 \text{ for } 0 < x < 125 \qquad \text{and} \qquad f'(x) < 0 \text{ for } 125 < x.$$

Thus $x = 125$ really gives a global maximum value for $f(x)$ over all $x > 0$. So the company should use 125 hours of student labour each day.

Discussion: A number of writers assumed the company was so small that it could only hire one worker, and imposed the constraint $L \leq 24$. This cost them one or two marks, depending on the clarity with which they expressed their conclusions using the sign of f'.

[8] **13.** There are two lines that are simultaneously tangent to both the curves

$$y = x^2 \quad \text{and} \quad y = -4(x-1)^2.$$

Find the equations of these two lines.

Hint: Make a sketch. Find the equation of the tangent line to the first curve at $x = a$, and the equation of the tangent line to the second curve at $x = b$.

Solution: The curve $y = x^2$ has $y' = 2x$, so its tangent at $x = a$ is

$$y = a^2 + 2a(x - a) = 2ax - a^2.$$

The curve $y = -4(x-1)^2$ has $y' = -8(x-1)$, so its tangent at $x = b$ is

$$y = -4(b-1)^2 - 8(b-1)(x-b) = -8(b-1)x + 4b^2 - 4.$$

These lines are identical when they have the same slopes and the same y-intercepts, i.e., when a and b obey

$$(1) \qquad 2a = 8(1 - b),$$
$$(2) \qquad -a^2 = 4b^2 - 4.$$

Solve (1) for $a = 4(1 - b)$ and substitute into (2):

$$-16(b-1)^2 = 4(b-1)(b+1) \quad \Longleftrightarrow \quad 0 = 4(b-1)\left[b + 1 + 4(b-1)\right] = 4(b-1)[5b - 3].$$

This has two solutions: $b = 1$ leads to $a = 0$ and the simultaneous tangent

$$y = 0.$$

The second solution, $b = 3/5$, leads to $a = 8/5$ and the simultaneous tangent

$$y = \frac{16}{5}x - \frac{64}{25}.$$

Discussion: Following the hint and writing down two tangent lines would earn 3 marks; looking at the sketch and recognizing the simultaneous tangent $y = 0$ would earn another one. In spite of this, the most common score on this question was 1 mark, earned for calculating the derivatives $y' = 2x$ and $y' = -8(x-1)$. Most students then mistakenly equated these two values, apparently forgetting that $y'(x)$ gives the slope of the tangent line that touches the curve at x, and we are in a situation where the points of tangency for the two curves are different. This led them badly off-track and earned no further points. Among the writers who took the hint seriously, inflexibility with notation was an issue: many insisted on writing b for the y-intercept of one or both of the the tangent lines in question, creating a conflict with the meaning of the symbol b declared in the hint.

[8] **14.** The region L shown in the sketch below lies between the curves $y = x^p$ and $y = x^{1/p}$ for some constant $p > 1$. It is contained in the square where $0 \le x \le 1$, $0 \le y \le 1$.

Express the area of L in terms of p.
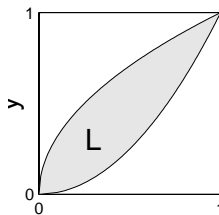
Solution: When $0 < x < 1$ and $p > 1$, we have $x^p < x < x^{1/p}$. Hence

$$
\begin{aligned}
\text{Area}(L) &= \int_0^1 \left(x^{1/p} - x^p \right) dx \\
&= \int_0^1 x^{1/p}\, dx - \int_0^1 x^p\, dx \\
&= \left[\frac{1}{(1/p) + 1} x^{(1/p)+1} \right]_{x=0}^1 - \frac{1}{p+1} x^{p+1} \Big|_{x=0}^1 \\
&= \frac{1}{(1/p) + 1} - \frac{1}{p+1} = \frac{p}{1+p} - \frac{1}{p+1} \\
&= \frac{p-1}{p+1}.
\end{aligned}
$$

Discussion: Writers who mis-identified which curve was higher but did everything else correctly (finishing with an area formula that was *negative* for all $p > 1$) should be encouraged to think about their conclusions, but lost only one point.

The End

This examination has 13 pages including this cover.

UBC-SFU-UVic-UNBC Calculus Examination

9 June 2005, 12:00-15:00

Name: ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯   **Signature:** ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

School: ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯   **Candidate Number:** ⎯⎯⎯⎯⎯⎯⎯⎯

Rules and Instructions

1. *Show all your work!* Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.

2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.

3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.

4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, *all calculator memories must be empty when the exam begins.*

5. If you need more space to solve a problem on page n, work on the back of page $n-1$.

6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:

   (a) Using any books, papers or memoranda.

   (b) Speaking or communicating with other candidates.

   (c) Exposing written papers to the view of other candidates.

7. Do not write in the grade box shown to the right.

1		5
2		5
3		5
4		5
5		8
6		8
7		8
8		8
9		8
10		8
11		8
12		8
13		8
14		8
Total		100