Short Key for 2004 Calculus Challenge Exam

Note: There is no attempt here to describe all possible correct answers. In many cases other approaches to a question could garner full marks.

For the examiners, apart from the accuracy of the answers, the crucial test is whether the student has made clear the principles and/or method being used and whether those principles and/or method are sound.

A longer key indicating alternative ways of attacking some of the problems will be posted later.

[3] **1.** Evaluate $\lim_{t\to 0} \dfrac{4-(t+2)^2}{t}$

ANSWER: -4

JUSTIFY YOUR ANSWER

Note that $\dfrac{4-(t+2)^2}{t} = -4-t$ for all $t \neq 0$. Therefore

$$\lim_{t\to 0} \frac{4-(t+2)^2}{t} = \lim_{t\to 0}\,(-4-t) = -(\lim_{t\to 0} 4) - (\lim_{t\to 0} t) = -4-0.$$

[4] **2.** Find a constant k such that

$$y = 2x - kx^2$$

is a solution of the differential equation $xy' = y - x^2$.

ANSWER: 1

JUSTIFY YOUR ANSWER

Substituting $y = 2x - kx^2$ in the two sides of the differential equation we get LHS $= x(2-2kx) = 2x - 2kx^2$, and RHS $= 2x - kx^2 - x^2 = 2x - (k+1)x^2$. Note that the LHS and the RHS are the same function of x if and only if $2k = k+1$, that is, if and only if $k = 1$. ∎

[6] **3.** Let $f(x) = 1 - x^2$.

Working directly from the definition of the derivative as a limit, verify the formula:

$$f'(a) = -2a$$

SHOW YOUR WORK

Note that $\dfrac{(1-x^2)-(1-a^2)}{x-a} = -a-x$ for all $x \neq a$. Therefore

$$\lim_{x\to a} \frac{(1-x^2)-(1-a^2)}{x-a} = \lim_{x\to a}\,(-a-x) = -(\lim_{x\to a} a) - (\lim_{x\to a} x) = -2a\,.$$

[4] **4.** Let $f(x)$ denote the function defined by $f(x) = \begin{cases} \dfrac{1-e^x}{x} & \text{if } x \neq 0 \\ -1 & \text{if } x = 0. \end{cases}$

Show that $f(x)$ is continuous at $x = 0$.

EXPLANATION

Using l'Hospital's rule and the continuity of e^x, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-e^x}{x} = \lim_{x\to 0} \frac{-e^x}{1} = -e^{\lim_{x\to 0} x} = -e^0 = -1.$$

By definition it follows that f is continuous at 0. ∎

[3] **5.** (a) Evaluate $\dfrac{d}{dx}\left(\dfrac{x^2-1}{x^2+1}\right)$ and simplify your answer.

ANSWER: $\dfrac{4x}{(x^2+1)^2}$

SHOW YOUR WORK Using the quotient rule, we have

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

[3] (b) Evaluate $\dfrac{d}{dx}\left[\tan\left(e^x+1\right)\right]$.

ANSWER: $e^x \sec^2\left(e^x+1\right)$

SHOW YOUR WORK: Using the rule for diferentiating composite functions, we have

$$\frac{d}{dx}\tan\left(e^x+1\right) = \sec^2\left(e^x+1\right)\frac{d}{dx}\left(e^x+1\right) = e^x\sec^2\left(e^x+1\right).$$

[3] (c) Given that

$$\sqrt{x^2+y^2} + \sqrt{xy} = 3\left(1+\sqrt{2}\right)$$

find an expression for dy/dx in terms of x, y.

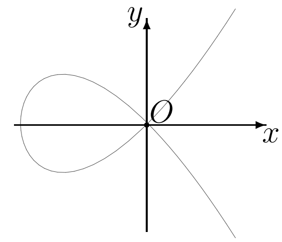
No simplification is necessary.

ANSWER: $\dfrac{-\left(\frac{x}{\sqrt{x^2+y^2}} + \frac{y}{2\sqrt{xy}}\right)}{\left(\frac{y}{\sqrt{x^2+y^2}} + \frac{x}{2\sqrt{xy}}\right)}$

SHOW YOUR WORK: Differentiating implicitly, we get

$$\frac{x+y(dy/dx)}{\sqrt{x^2+y^2}} + \frac{y+x(dy/dx)}{2\sqrt{xy}} = 0.$$

Solving for dy/dx we get the answer in the answer box. ∎



6. Consider the curve $\mathcal{C}$ described by the equation:

$$y^2 = x^3 + x^2$$

[4] (a) Find the coordinates of the points of $\mathcal{C}$ at which the tangent lines are parallel to the x-axis.

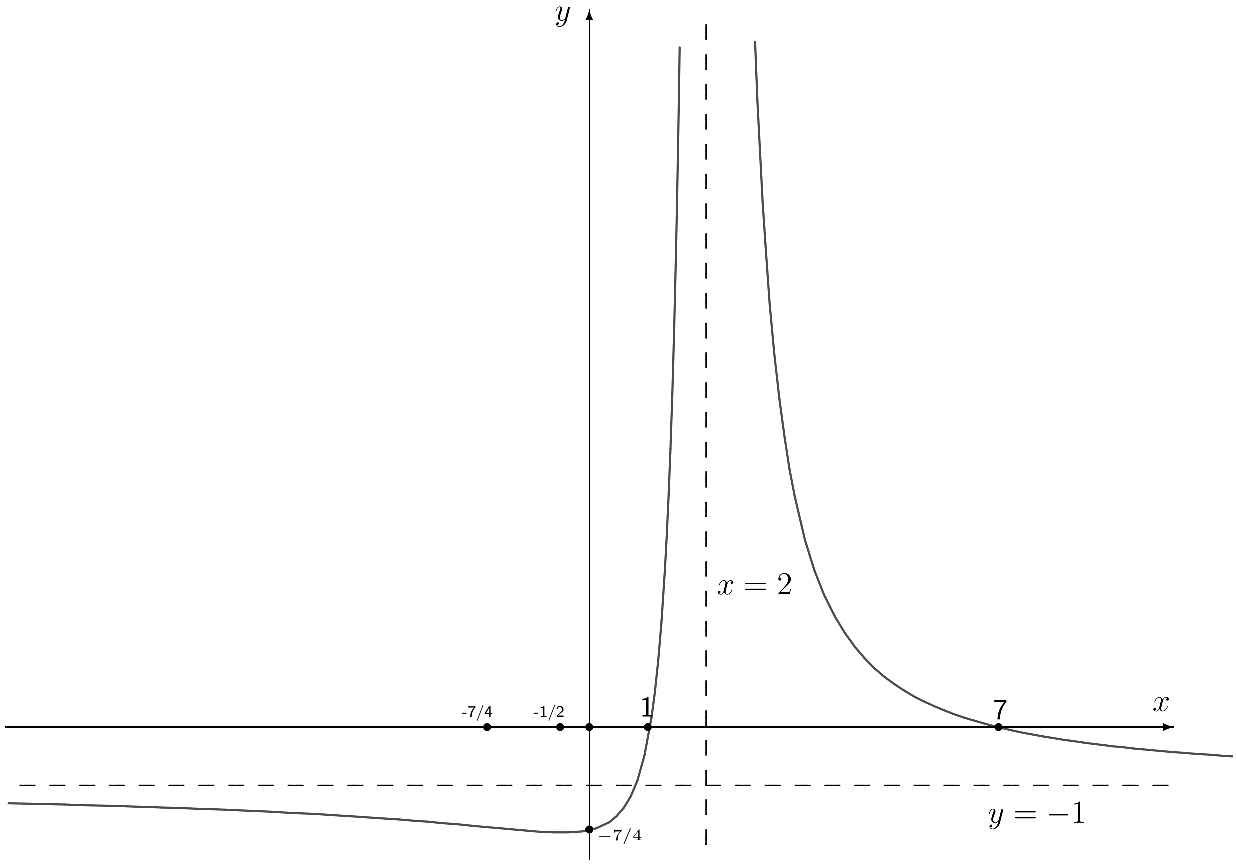
ANSWER: $\left(-2/3, \pm 2/(3\sqrt{3})\right)$

SHOW YOUR WORK: Differentiating we get $2y(dy/dx) = 3x^2 + 2x$. So for $dy/dx = 0$ we need either $x = 0$ or $x = -2/3$. Now, as we see in part (b), $x = 0$ does not imply $dy/dx = 0$. However, $x = -2/3$ *does* because the corresponding values of y, $y = \pm 2/(3\sqrt{3})$, are nonzero.

[4] (b) Find the equations of the tangents to $\mathcal{C}$ at the origin $(0, 0)$ and justify your answer.

ANSWER: $y = \pm x$

JUSTIFICATION: Taking square roots see that near $x = 0$ the curve consists of two parts $y = x\sqrt{x+1}$ and $y = -x\sqrt{x+1}$. The first branch of the curve has tangent $y = x$ at $x = 0$. The second branch of the curve has tangent $y = -x$ at $x = 0$. ■



[6] **7.** The following information is given about the function f:

$$f(x),\ f'(x),\ f''(x) \text{ are defined and continuous for all } x \neq 2$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = -1, \quad \lim_{x\to 2} f(x) = \infty$$

$$f(1) = f(7) = 0 \quad f'(-1/2) = 0 \quad f(0) = -7/4$$

$$1,\ 7 \text{ are the only zeros of } f(x);\ -1/2 \text{ is the only zero of } f'(x)$$

$$f''(x) < 0 \text{ for all } x \in (-\infty, -7/4),\ f''(-7/4) = 0$$

$$f''(x) > 0 \text{ for all } x \text{ in } (-7/4, 2) \text{ and all } x \text{ in } (2, \infty)$$

Using the axes provided above sketch the graph of $y = f(x)$ in a manner consistent with all the information.

No justification is required, but you may add comments if you wish.

[6] **8.** (a) The sun is directly overhead at noon and sets at 6 p.m. Assuming that θ, the angle of elevation of the sun above the horizon, changes at a constant rate, show that, at 4 p.m., the length of the shadow cast by a 6 metre post is increasing at a rate of $\pi/30$ metres per minute.

EXPLANATION: The angle of θ of elevation is $\pi/2$ at noon and 0 at 6 p.m. Therefore $d\theta/dt = -(\pi/2)/360 = -\pi/720$, and at 4 p.m. $\theta = \pi/6$.

The length of the shadow is $L = 6\cot\theta$ metres. So its rate of increase is $dL/dt = -6\,\text{cosec}^2\theta(d\theta/dt)$ $= -6(2^2)(-\pi/720) = \pi/30$ metres per minute. ■

[6] (b) The pressure P, volume V, and temperature T of the gas in a spherical balloon of radius r are related by the universal gas equation

$$PV = nRT$$

where n is the number of moles of gas, and R is a constant. Here the temperature T is measured in Kelvins.

Let t be the elapsed time in hours. A variable x is said to be *increasing at a% per hour* if $\frac{1}{x}\frac{dx}{dt} = \frac{a}{100}$.

At the instant under consideration n is not changing, the temperature of the gas is increasing at 4% per hour, and the pressure of the gas is increasing at 1% per hour. Show that r, the radius of the balloon is also increasing at 1% per hour.

EXPLANATION: Let t denote elapsed time in hours. Taking natural logarithms,

$$\ln P + \ln V = \ln(nR) + \ln T.$$

Since nR is constant, differentiating with respect to t, we get $\frac{1}{P}\frac{dP}{dt} + \frac{1}{V}\frac{dV}{dt} = 0 + \frac{1}{T}\frac{dT}{dt}$. It follows that $\frac{1}{V}\frac{dV}{dt} = \frac{3}{100}$. From $V = (4\pi r^3)/3$ it follows that $\frac{1}{V}\frac{dV}{dt} = \frac{3}{r}\frac{dr}{dt}$. So $\frac{1}{r}\frac{dr}{dt} = \frac{1}{100}$. ■

[4] **9.** (a) Find the linear approximation of the function $\sqrt[3]{x}$ at $x = 1000$ and use it to approximate $\sqrt[3]{1002}$.

ANSWER:

linearization: 10 + (1/300)(x-1000)

$\sqrt[3]{1002} \approx 10 + (2/300)$

SHOW YOUR WORK: The linearization of $f(x)$ at $x = a$ is $f(a) + (x-a)f'(a)$. Here $a = 1000$ and $f'(a) = (1/3)a^{-2/3} = 1/300$.

So $f(1002) \approx f(1000) + (2)f'(1000) = 10 + (2/300)$. ■

[4] (b) Beginning with the initial estimate $x_0 = 10$ apply one step of Newton's Method to find an estimate for a root of $x^3 - 1002 = 0$.

ANSWER: $x_1 = 10 + (2/300)$

SHOW YOUR WORK: Given initial estimate $x_0 = a$, the next estimate to a root of $f(x) = 0$ is $x_1 = a - \dfrac{f(a)}{f'(a)}$. Take $f(x) = x^3 - 1002$ and $x_0 = a = 10$. Then $f(a) = -2$ and $f'(a) = 3a^2 = 300$. So $x_1 = 10 + (2/300)$. ∎

10. Let $f(x) = \frac{1}{2}x^4 - x^3 - 6x^2 + 4x$.

$x = a$ is called a *critical point of* $f(x)$ if $f'(x) = 0$.

[2] (a) Find all the critical points of $f(x)$ given that one of them is $x = -2$.

ANSWER: $-2,\ \dfrac{7 \pm \sqrt{33}}{4}$

SHOW YOUR WORK: Note that $f'(x) = 2x^3 - 3x^2 - 12x + 4$. Since -2 is given as a root we see that $f'(x) = (x+2)(2x^2 - 7x + 2)$. Solving the quadratic we get the roots:

$$x = -2,\ \frac{7 \pm \sqrt{33}}{4}$$

[3] (b) At which values of x, if any, does $f(x)$ have a local maximum?

At which values of x, if any, does $f(x)$ have a local minimum?

ANSWER:

local maximum(s): $[7 - \sqrt{33}]/4$

local minimum(s): $-2,\ [7 + \sqrt{33}]/4$

EXPLAIN: One can use the second-derivative test. However, let us look at the sign changes of $f'(x)$ instead. Let $\alpha_1 = -2$, $\alpha_2 = (7 - \sqrt{33})/4$, and $\alpha_3 = (7 + \sqrt{33})/4$. Then α_1, α_2, α_3 are the roots of $f'(x)$ in increasing order, and $f(x) = 2(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$. As x increases through α_1 and α_3, $f'(x)$ changes from negative to positive. So $f(x)$ has local minimums at $x = \alpha_1$, α_3. As x increases through α_2, $f'(x)$ changes from positive to negative. So $f(x)$ has a local maximum at $x = \alpha_2$. ∎

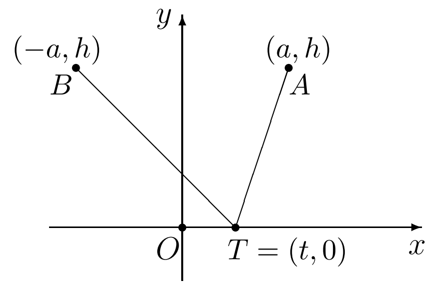
[3] (c) What is the largest interval on which $f(x)$ is concave down?

ANSWER: $[-1, 2]$

EXPLAIN: An easy calculation shows that $f''(x) = 6(x+1)(x-2)$. "Concave down" means that $f'(x)$ is decreasing. Now $f'(x)$ is decreasing on $[-1, 2]$ because $f''(x)$ is negative on $(-1, 2)$. Also, any interval which contains a point not in $[-1, 2]$ will contain a subinterval on which $f'(x)$ is increasing. So $[-1, 2]$ contains *all* intervals on which $f(x)$ is concave down. ∎

11. The point $T = (t, 0)$ varies on the x-axis. The points $A = (a, h)$ and $B = (-a, h)$ are fixed with $a,\ h > 0$. Define the function L by

$$L(t) = \text{length}(AT) + \text{length}(BT).$$



[4] (a) Express $L(t)$ in terms of t and the constants $a,\ h$.

ANSWER:
$$L(t) = \sqrt{h^2 + (t-a)^2} + \sqrt{h^2 + (t+a)^2}$$

EXPLANATION: Using the formula for the distance between two points in the plane, the first term is the length of AT and the second is the length of BT. ■

[5] (b) Use calculus to show that $L(t)$ has an absolute minimum when $t = 0$.

EXPLANATION: Note that

$$\frac{dL}{dt} = \frac{t-a}{\sqrt{h^2 + (t-a)^2}} + \frac{t+a}{\sqrt{h^2 + (t+a)^2}}.$$

Therefore $dL/dt = 0$ when $t = 0$. Differentiating again we get

$$\frac{d^2L}{dt^2} = \frac{h^2}{\left(h^2 + (t-a)^2\right)^{3/2}} + \frac{h^2}{\left(h^2 + (t+a)^2\right)^{3/2}} > 0 \qquad (-\infty < t < \infty).$$

So dL/dt is increasing on $(-\infty, \infty)$. It follows that $dL/dt < 0$ for all $t < 0$ and $dL/dt > 0$ for all $t > 0$. Thus $L(t)$ is decreasng on $(-\infty, 0]$, and increasing on $[0, \infty)$. This is enough. ■

[4] **12.** (a) Find the general antiderivative of $\sin 2x + \dfrac{2}{x^2}$.

ANSWER:
$$-\frac{1}{2}\cos(2x) - \frac{2}{x}$$

SHOW YOUR WORK: Here we are exploiting the basic differentiation rules: $(d/dx)(\cos x) = -\sin x$ and $(d/dx)(1/x) = -1/x^2$. ■

[3] (b) Find a function y defined on $(0, \infty)$ such that

$$\frac{dy}{dx} = \frac{2x^2 + 1}{x}, \quad y(1) = 0.$$

ANSWER:
$$y = x^2 + \ln x - 1$$

SHOW YOUR WORK: Taking antiderivatives in the differential equation we get $y = x^2 + \ln x + C$. To satisfy the condition $y(1) = 0$ we take $C = -1$. ■

13. A tank of brine has 1000 litre capacity and initially contains 50 kilograms of salt dissolved in water.

Brine is drawn from the tank at rate of 5 litres per minute and water is added to the tank at the same rate to maintain the volume of solution at 1000 litres.

The tank is well-stirred so that the concentration of salt is uniform at all times.

Let S denote the amount of salt (in kilograms) in the tank after t minutes.

[2] (a) What is the approximate net change ΔS in the amount of salt in the tank in the time interval $[t, t + \Delta t]$ if Δt is small?

Write your answer as a constant multiple of $S\Delta t$.

ANSWER: $\approx -(1/200)S\Delta t$

SHOW YOUR WORK: In Δt minutes the proportion of the solution which is drawn from the tank is $(5\Delta t)/1000$. So the net change is $-(5S\Delta t)/1000$. ■

[2] (b) Write down an equation relating dS/dt and S, where t is the elapsed time in minutes.

ANSWER: $\dfrac{dS}{dt} = -S/200$

SHOW YOUR WORK: From (a), $\Delta S \approx -(1/200)S\Delta t$. Dividing by Δt, we have

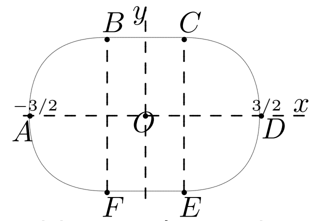
$$\frac{\Delta S}{\Delta t} \approx -(1/200)S.$$

Taking the limit as $\Delta t \to 0$, we have $dS/dt = -S/200$. ■

[4] (c) How many minutes pass before there are only 25 kilograms of salt in the tank?

ANSWER: $t = 200 \ln 2$

SHOW YOUR WORK: Writing the equation as $(1/S)dS/dt = -1/200$ and taking antiderivatives with respect to t we get $\ln S = -t/200 + C$. Rewriting this we get $S = ke^{-t/200}$, where k is constant. At $t = 0$, $S = 50$. So $k = 50$. For $S = 25$ we need $e^{-t/200} = 1/2$. Solving we get $t = 200 \ln 2$. ■



[6] **14.** An oval plate is symmetric about its axes, which are shown as Ox, Oy in the figure. The midsection of the plate is a rectangle $BCEF$ of width 1 and height 2.

The arc AB of the bounding curve has the same shape as the arc $y = 2\sqrt{x} - x \quad (0 \le x \le 1)$. Indeed, the arc AB is obtained by translating the arc $y = 2\sqrt{x} - x \quad (0 \le x \le 1)$ horizontally $3/2$ units to the left.

Show the area of the plate is $16/3$.

EXPLANATION: The basic principle is that, if $a < b$ and $f(x)$ is continuous on $[a, b]$, then the area 'under $y = f(x)$ from $x = a$ to $x = b$' is equal to $F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$.

Note that $\int (2\sqrt{x} - x)\,dx = (4/3)x^{3/2} - (1/2)x^2 + C$. Hence the area under arc AB and above the x-axis is $(4/3) - (1/2) = 5/6$. So total area of the plate is $4 \cdot (5/6) + \text{area(BCEF)} = 16/3$. ■