SFU - UBC - UNBC - UVic

Calculus Challenge Exam

June 3, 2004, 12 noon to 3 p.m.

Host: SIMON FRASER UNIVERSITY

Student signature

INSTRUCTIONS

1. Show all your work. Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are calculator ready, like $3 + \ln 7$ or e^2, are preferred.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
5. If you need more space to solve a problem on page n, work on the back of page n - 1.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.

Question	Maximum	Score
1	3	
2	4	
3	6	
4	4	
5	11	
6	8	
7	6	
8	12	
9	8	
10	8	
11	9	
12	7	
13	8	
14	6	
Total	100	

[3] **1.** Evaluate $\lim_{t \to 0} \dfrac{4-(t+2)^2}{t}$.

ANSWER:

JUSTIFY YOUR ANSWER

[4] **2.** Find a constant k such that

$$y = 2x - kx^2$$

is a solution of the differential equation $xy' = y - x^2$.

ANSWER:

JUSTIFY YOUR ANSWER

[6] **3.** Let $f(x) = 1 - x^2$.

Working directly from the definition of the derivative as a limit, verify the formula:

$$f'(a) = -2a$$

VERIFICATION

[4] **4.** Let $f(x)$ denote the function defined by $f(x) = \begin{cases} \dfrac{1-e^x}{x} & \text{if } x \neq 0 \\ -1 & \text{if } x = 0. \end{cases}$

Show that $f(x)$ is continuous at $x = 0$.

EXPLANATION

[3] **5.** (a) Evaluate $\dfrac{d}{dx}\left(\dfrac{x^2-1}{x^2+1}\right)$ and simplify your answer.

ANSWER:

SHOW YOUR WORK

[3] (b) Evaluate $\dfrac{d}{dx}\left[\tan\left(e^x+1\right)\right]$.

ANSWER:

SHOW YOUR WORK

[5] (c) Given that

$$\sqrt{x^2 + y^2} + \sqrt{xy} = 1$$

find an expression for dy/dx in terms of x, y.

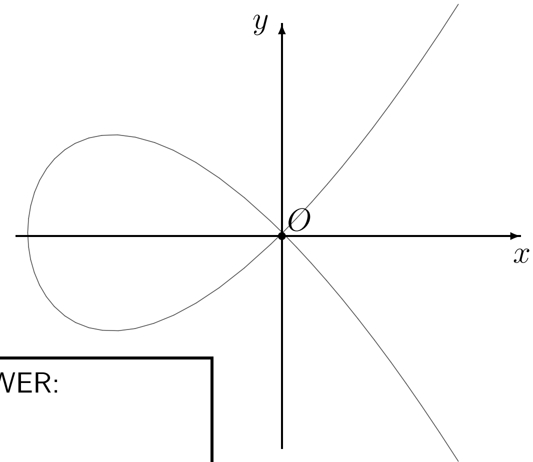
No simplification is necessary.

ANSWER:

SHOW YOUR WORK

6. Consider the curve $\mathcal{C}$ described by the equation:

$$y^2 = x^3 + x^2$$



[4] (a) Find the coordinates of the points of $\mathcal{C}$ at which the tangent lines are parallel to the x-axis.

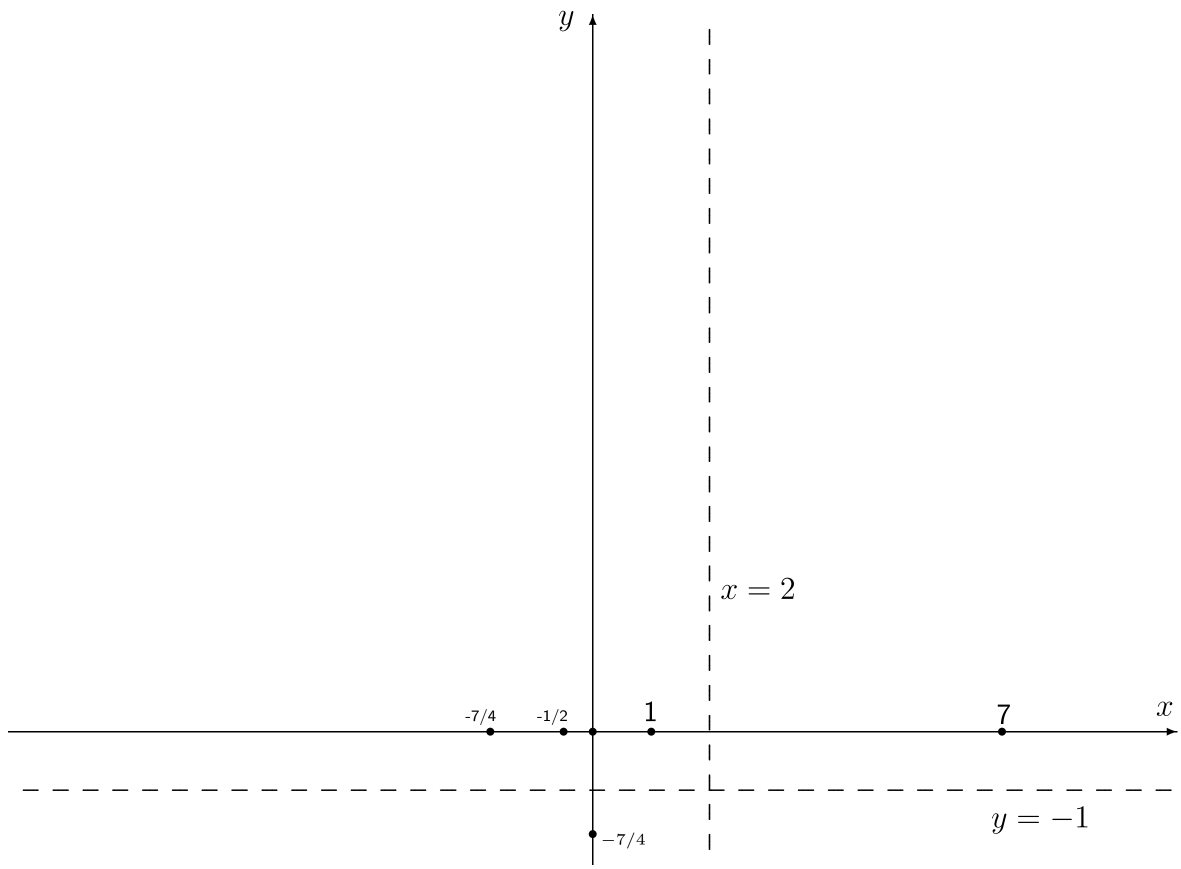
ANSWER:

SHOW YOUR WORK

[4] (b) Find the equations of the tangents to $\mathcal{C}$ at the origin $(0, 0)$ and justify your answer.

ANSWER:

JUSTIFICATION



[6] **7.** The following information is given about the function f:

$$f(x),\ f'(x),\ f''(x) \text{ are defined and continuous for all } x \neq 2$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = -1, \quad \lim_{x\to 2} f(x) = \infty$$

$$f(1) = f(7) = 0 \quad f'(-1/2) = 0 \quad f(0) = -7/4$$

$$1,\ 7 \text{ are the only zeros of } f(x);\ -1/2 \text{ is the only zero of } f'(x)$$

$$f''(x) < 0 \text{ for all } x \in (-\infty, -7/4),\ f''(-7/4) = 0$$

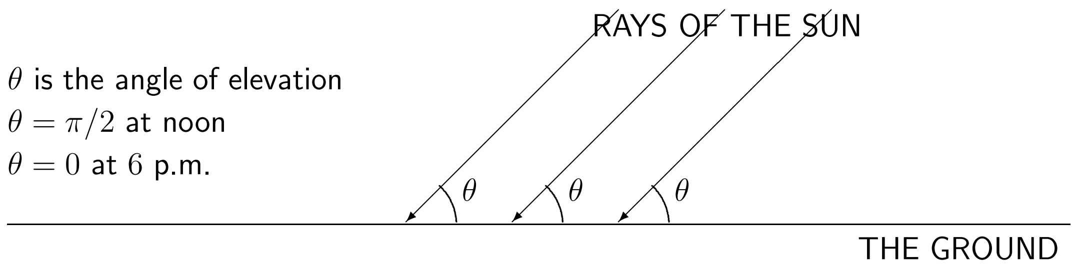
$$f''(x) > 0 \text{ for all } x \text{ in } (-7/4, 2) \text{ and all } x \text{ in } (2, \infty)$$

Using the axes provided above sketch the graph of $y = f(x)$ in a manner consistent with all the information.

No justification is required, but you may add comments if you wish.

[6] **8.** (a) The sun is directly overhead at noon and sets at 6 p.m. Assuming that θ, the angle of elevation of the sun above the horizon, changes at a constant rate, show that at 4 p.m. the length of the shadow cast by a 6 metre high post is increasing at a rate of $\pi/30$ metres per minute.

θ is the angle of elevation
$\theta = \pi/2$ at noon
$\theta = 0$ at 6 p.m.



EXPLANATION

[6] (b) The pressure P, volume V, and temperature T of the gas in a spherical balloon of radius r are related by the universal gas equation

$$PV = nRT$$

where n is the number of moles of gas, and R is a constant. Here the temperature T is measured in Kelvins.

Let t be the elapsed time in hours. A variable x is said to be *increasing at a% per hour* if $\frac{1}{x}\frac{dx}{dt} = \frac{a}{100}$.

At the instant under consideration n is not changing, the temperature of the gas is increasing at 4% per hour, and the pressure of the gas is increasing at 1% per hour. Show that r, the radius of the balloon is also increasing at 1% per hour.

EXPLANATION

[4] **9.** (a) Find the linear approximation of the function $\sqrt[3]{x}$ at $x = 1000$ and use it to approximate $\sqrt[3]{1002}$.

ANSWER:

linearization:

$\sqrt[3]{1002} \approx$

SHOW YOUR WORK

[4] (b) Beginning with the initial estimate $x_0 = 10$ apply one step of Newton's Method to find an estimate for a root of $x^3 - 1002 = 0$.

ANSWER:

$x_1 =$

SHOW YOUR WORK

10. Let $f(x) = \frac{1}{2}x^4 - x^3 - 6x^2 + 4x$.

$x = a$ is called a *critical point of* $f(x)$ if $f'(x) = 0$.

[2] (a) Find all the critical points of $f(x)$ given that one of them is $x = -2$.

ANSWER:

SHOW YOUR WORK

[3] (b) At which values of x, if any, does $f(x)$ have a local maximum?

At which values of x, if any, does $f(x)$ have a local minimum?

ANSWER:

local maximum(s):

local minimum(s):

EXPLAIN

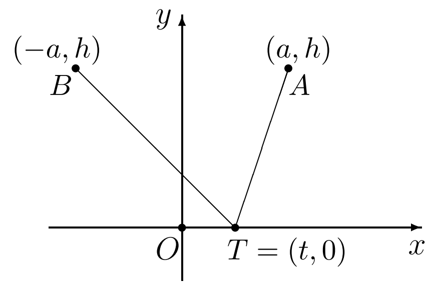
[3] (c) What is the largest interval on which $f(x)$ is concave down?

ANSWER:

EXPLAIN:

11. The point $T = (t, 0)$ varies on the x-axis. The points $A = (a, h)$ and $B = (-a, h)$ are fixed with a, $h > 0$. Define the function L by

$$L(t) = \text{length}(AT) + \text{length}(BT).$$



[4] (a) Express $L(t)$ in terms of t and the constants a, h.

ANSWER:

EXPLANATION

[5] (b) Use calculus to show that $L(t)$ has an absolute minimum when $t = 0$.

EXPLANATION

[4] **12.** (a) Find the general antiderivative of $\sin 2x + \dfrac{2}{x^2}$.

ANSWER:

SHOW YOUR WORK

[3] (b) Find a function y defined on $(0, \infty)$ such that

$$\frac{dy}{dx} = \frac{2x^2 + 1}{x}, \;\; y(1) = 0.$$

ANSWER:

SHOW YOUR WORK

13. A tank of brine has 1000 litre capacity and initially contains 50 kilograms of salt dissolved in water.

Brine is drawn from the tank at rate of 5 litres per minute and water is added to the tank at the same rate to maintain the volume of solution at 1000 litres.

The tank is well-stirred so that the concentration of salt is uniform at all times.

Let S denote the amount of salt (in kilograms) in the tank after t minutes.

[2] (a) What is the approximate net change ΔS in the amount of salt in the tank in the time interval $[t, t + \Delta t]$ if Δt is small?

Write your answer as a constant multiple of $S\Delta t$.

ANSWER:

SHOW YOUR WORK

[2] (b) Write down an equation relating dS/dt and S.

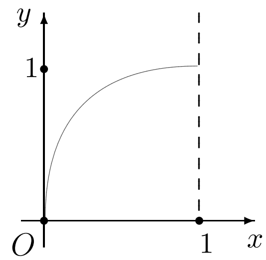
ANSWER:

SHOW YOUR WORK

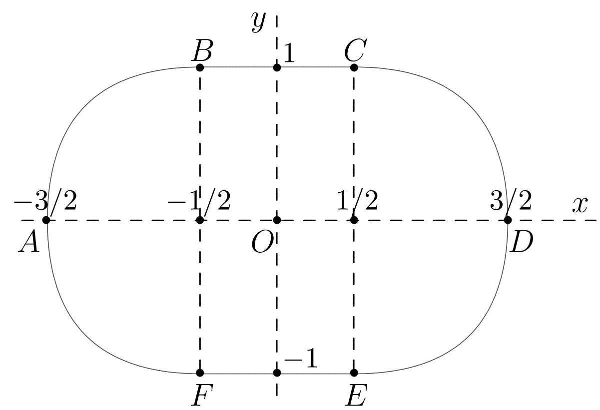
[4] (c) How many minutes pass before there are only 25 kilograms of salt in the tank?

ANSWER:

SHOW YOUR WORK



Graph of $y = 2\sqrt{x} - x$



[6] **14.** An oval plate is symmetric about its axes, which are shown as Ox, Oy in the figure.

The midsection of the plate is a rectangle $BCEF$ of width 1 and height 2.

The arc AB of the bounding curve has the same shape as the arc

$$y = 2\sqrt{x} - x \quad (0 \le x \le 1).$$

Indeed, the arc AB is obtained by translating the arc $y = 2\sqrt{x} - x \quad (0 \le x \le 1)$ horizontally $3/2$ units to the left.

Show that the area of the plate is $16/3$.

EXPLANATION