Key for the 2002 Calculus Challenge Exam

Note: There is no attempt here to describe all possible correct answers. Students sitting the calculus challenge examination will have used a variety of texts and been exposed to a variety of teaching styles.

For the examiners, apart from the accuracy of the answers, the crucial test is whether the student has made clear the principles and/or method being used and whether those principles and/or method are sound.

Marks are not deducted for sufficiently trivial errors, e.g., inadvertently dropping a sign.

1. Compute the following limits.

[3] (a) $\displaystyle\lim_{t\to -2} \frac{t^2-t-6}{t^2+5t+6}$

ANSWER: -5

JUSTIFY YOUR ANSWER

Note that $\dfrac{t^2-t-6}{t^2+5t+6} = \dfrac{(t+2)(t-3)}{(t+2)(t+3)} = \dfrac{t-3}{t+3}$ whenever $t \neq -2$. Therefore

$$\lim_{t\to -2} \frac{t^2-t-6}{t^2+5t+6} = \lim_{t\to -2} \frac{t-3}{t+3} = \frac{(\lim_{t\to -2} t) - 3}{(\lim_{t\to -2} t) + 3} = \frac{-2-3}{-2+3} = -5.$$

Instead of using the limit laws to evaluate $\displaystyle\lim_{t\to -2} \frac{t-3}{t+3}$, one may use the fact that a rational function is continuous, i.e., continuous at each point of its domain. ∎

[3] (b) $\displaystyle\lim_{x\to 0^+} \left[\frac{1}{x}\left(3e^{-1/x} + 5e^x\right)\sin x\right]$

ANSWER: 5

JUSTIFY YOUR ANSWER

From the limit laws the given limit is equal to

$$\left(\lim_{x\to 0^+} \frac{\sin x}{x}\right)\left(3\lim_{x\to 0^+} e^{-1/x} + 5\lim_{x\to 0^+} e^x\right)$$

provided the three limits in the line above exist. We may take $\lim_{x\to 0^+}(\sin x)/x = 1$ as known. (A second way of looking at this limit is via l'Hospital's rule. A third way is to notice that, since $(d/dx)\sin x = \cos x$, we have $\lim_{h\to 0}(\sin h)/h = \cos 0 = 1$.) Now e^x is continuous so $\lim_{x\to 0^+} e^x = e^0 = 1$. Finally, as $x \to 0^+$, $1/x \to \infty$, whence $e^{1/x} \to \infty$ and so $e^{-1/x} \to 0$. Thus the given limit evaluates to $1 \cdot (0+5) = 5$. ∎

[4] **2.** (a) Find the asymptotes of $y = \left(\dfrac{x}{x-1}\right)^2$ and justify your answer.

ANSWER:
$y = 1$
$x = 1$

EXPLANATION

The following is an acceptable explanation: $[x/(x-1)]^2$ is defined except at $x = 1$. Also,

$$\lim_{x\to\infty}\left(\frac{x}{x-1}\right)^2 = \lim_{x\to-\infty}\left(\frac{x}{x-1}\right)^2 = 1 \text{ and } \lim_{x\to1^+}\left(\frac{x}{x-1}\right)^2 = \lim_{x\to1^-}\left(\frac{x}{x-1}\right)^2 = \infty. \qquad \blacksquare$$

[2] (b) Where does the curve $y = \left(\dfrac{x}{x-1}\right)^2$ cross its horizontal aymptote?

ANSWER:
$(1/2, 1)$

EXPLANATION

We have to solve the equations: $y = 1$ and $y = [x/(x-1)]^2$. Eliminating y, we have $x = \pm(x-1)$. The only solution is $x = 1/2$, which gives $y = 1$. ■

[4] **3.** (a) Find $\dfrac{dv}{du}$ when $v = \sqrt{\dfrac{\tan u}{1+\tan u}}$.

ANSWER:
$$\frac{dv}{du} = \frac{\sec^2 u}{2\,(1+\tan u)^{3/2}\,(\tan u)^{1/2}}$$

SHOW YOUR WORK

Using the chain rule and the quotient rule, we have:

$$\frac{dv}{du} = \frac{1}{2}\sqrt{\frac{1+\tan u}{\tan u}}\,\frac{d}{du}\left(\frac{\tan u}{1+\tan u}\right) = \frac{1}{2}\,\frac{(1+\tan u)^{1/2}}{(\tan u)^{1/2}}\,\frac{(1+\tan u)\sec^2 u - \tan u \sec^2 u}{(1+\tan u)^2} \qquad \blacksquare$$

[4] (b) Let a be a constant and $f(x) = \sin(ax)$. Find the 97-th derivative, $f^{(97)}(x)$, of the function $f(x)$.

ANSWER:
$a^{97}\cos ax$

SHOW YOUR WORK

We have: $f(x) = \sin(ax)$, $f'(x) = a\cos(ax)$, $f''(x) = -a^2\sin(ax)$, $f^{(3)}(x) = -a^3\cos(ax)$, $f^{(4)}(x) = a^4\sin(ax)$, ... There is a clear pattern from which we deduce $f^{(96)}(x) = a^{96}\sin(ax)$. ■

[3] **4.** (a) Find the general antiderivative of $\left(9-4x^2\right)^{-1/2}$.

ANSWER:
$$\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+C$$

SHOW YOUR WORK: Using the substitution $u=(2x)/3$, we have:

$$\int\left(9-4x^2\right)^{-1/2}\,dx=\int\left(9-4(3u/2)^2\right)^{-1/2}(3/2)du=\frac{1}{2}\int\left(1-u^2\right)^{1/2}\,du=\frac{1}{2}\sin^{-1}u+C \quad \blacksquare$$

[3] (b) It is given that

$$f'(x)=2^x+x^2 \text{ and } f(0)=0.$$

Find $f(x)$.

ANSWER:
$$f(x)=\frac{2^x-1}{\ln 2}+\frac{x^3}{3}$$

SHOW YOUR WORK: Writing 2^x as $e^{x\ln 2}$ we see that the antiderivative of 2^x+x^2 is $(e^{x\ln 2}/\ln 2)+(x^3)/3$. By the evaluation theorem,

$$f(x)-f(0)=\int_0^x f'(t)\,dt=\left[\frac{2^t}{\ln 2}+\frac{t^3}{3}\right]_0^x=\frac{2^x-1}{\ln 2}+\frac{x^3}{3}. \quad \blacksquare$$

[6] **5.** Use the definition of derivative (and <u>not</u> the product rule) to show that, if $f(x)$ is differentiable at $x=c$ and $g(x)=xf(x)$, then $g'(c)$ exists and $g'(c)=f(c)+cf'(c)$.

ANSWER: Using the definition of derivative we have

$$\begin{aligned} g'(c) &= \lim_{h\to 0}\frac{g(c+h)-g(c)}{h}=\lim_{h\to 0}\frac{(c+h)f(c+h)-cf(c)}{h} \\ &= \lim_{h\to 0}\left[c\left(\frac{f(c+h)-f(c)}{h}\right)+f(c+h)\right]=c\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\right)+\lim_{h\to 0}f(c+h) \\ &= cf'(c)+f(c) \end{aligned}$$

Note that $\lim_{h\to 0}f(c+h)=f(c)$ because differentiability of f at c (which is assumed) implies continuity at c. $\blacksquare$

[3] **6.** For what value of k is the function

$$h(x)=\begin{cases} 2x+3 & \text{if } x\le 1 \\ k-1 & \text{if } x>1 \end{cases}$$

continuous?

ANSWER:
$k=6$

JUSTIFY YOUR ANSWER: Since $2x+3$ and $k-1$ are continuous functions, $h(x)$ is continuous at $x=c$ for all $c\in(-\infty,1)\cup(1,\infty)$ and continuous on the left at $x=1$. Further, $\lim_{x\to 1^+}h(x)=k-1$ is equal to $h(1)=5$ if and only if $k=6$. $\blacksquare$

[4] **7.** (a) Express $\dfrac{dy}{dx}$ as a function of x, when $y = \left(\dfrac{x^7 \cos x}{7^x\sqrt{1+x^2}}\right)$.

ANSWER:

$$\frac{dy}{dx} = \left(\frac{x^7 \cos x}{7^x\sqrt{1+x^2}}\right)\left(\frac{7}{x} - \tan x - \ln 7 - \frac{x}{1+x^2}\right)$$

SHOW YOUR WORK: Taking natural logarithms and differentiating, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left(7\ln x + \ln\cos x - x\ln 7 - \frac{1}{2}\ln(1+x^2)\right) = \frac{7}{x} - \tan x - \ln 7 - \frac{x}{1+x^2}\,.$$

Although the line above is only valid for values of x such that $x, \cos x > 0$, the resulting formula is valid for all $x \neq 0$ such that $\tan x$ is defined. ■

[5] (b) Express $\dfrac{dy}{dx}$ as a function of x, when $y = x^{\ln x}$.

ANSWER:

$$\frac{dy}{dx} = 2(\ln x)x^{(\ln x)-1}$$

SHOW YOUR WORK: We have

$$\frac{dy}{dx} = \frac{d}{dx}\left((e^{\ln x})^{\ln x}\right) = \frac{d}{dx}\left(e^{(\ln x)^2}\right) = e^{(\ln x)^2}\frac{d}{dx}(\ln x)^2 = 2(\ln x)x^{(\ln x)-1}.$$ ■

8. A curve has the equation $\sin(x+y) = xe^y$.

[2] (a) Show that $(0, \pi)$ is on the curve.

ANSWER: We just observe that $\sin(0+\pi) = 0 = 0 \cdot e^{\pi}$. ■

[4] (b) Find the equation of the line tangent to the curve at $(0, \pi)$.

ANSWER:

$$y + (1 + e^{\pi})x = \pi$$

SHOW YOUR WORK: By implicit differentiation,

$$\cos(x+y)\left(1 + \frac{dy}{dx}\right) = e^y + xe^y\frac{dy}{dx}.$$

It follows that $\left(\frac{dy}{dx}\right)_{x=0, y=\pi} = -1 - e^{\pi}$. This allows us to write down the equation of the tangent using the point-slope form of the equation of a line. ■

[4] (c) A point moves along the curve so that at $(0, \pi)$ its x-coordinate is increasing at a rate of 3 units/sec. How fast is its y-coordinate changing at $(0, \pi)$?

ANSWER:

decreasing by $3(1 + e^{\pi})$ units per sec

SHOW YOUR WORK: In general, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. Thus

$$\left(\frac{dy}{dt}\right)_{x=0, y=\pi} = (-1 - e^{\pi})\left(\frac{dx}{dt}\right)_{x=0, y=\pi} = -3\left(1 + e^{\pi}\right).$$ ■

9. Let $f(x) = e^{x-2} + x^3 - 2$.

[3] (a) Use the derivative of f to explain why the equation $f(x) = 0$ has at most one solution.

EXPLANATION: Since $f'(x) = e^{x-2} + 3x^2 > 0$ for all x, the function f is strictly increasing. ■

[3] (b) Explain why $f(x) = 0$ has a solution in the interval $(1, 2)$.

EXPLANATION: Note that f is continuous, $f(1) = 1 - (1/e) < 0$, and $f(2) = 1 + 8 - 2 = 7 > 0$. By the intermediate value theorem, f has a zero in $(1, 2)$. ■

[3] (c) Newton's method with an initial estimate of 2 is used to find an approximate value for the solution of $f(x) = 0$.

What is the next estimate?

ANSWER: $19/13$

SHOW YOUR WORK: The next estimate is $2 - (f(2)/f'(2)) = 2 - (7/13) = 19/13$. This is the x-coordinate of the point in which the tangent to $y = f(x)$ at $(2, f(2))$ meets $y = 0$. ■

[6] **10.** A particle moves along the x-axis with velocity $\dfrac{1}{1+t^2}$ at time t. If it passes the point $\pi/6$ at time $t = 1$, what is its acceleration when it passes the point $\pi/4$?

ANSWER: $-\sqrt{3}/8$

SHOW YOUR WORK: We are given $\dfrac{dx}{dt} = \dfrac{1}{1+t^2}$. Taking antiderivatives, we get $x = \tan^{-1} t + C$, where C is a constant. Since $x(1) = \pi/6$, we see that $\pi/6 = (\pi/4) + C$. So $C = -\pi/12$. Letting $x = \pi/4$, we get $\pi/4 = \tan^{-1} t - (\pi/12)$. So $\tan^{-1} t = \pi/3$, which means $t = \sqrt{3}$ when $x = \pi/4$. The acceleration is given by

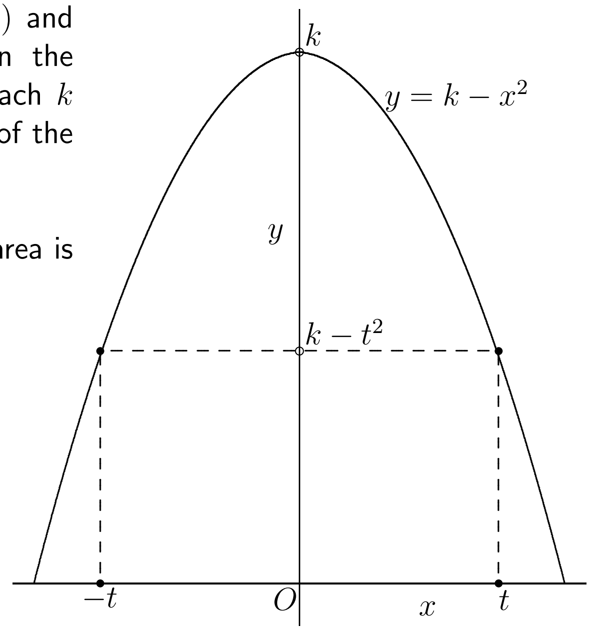
$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{1}{1+t^2}\right) = \frac{-2t}{(1+t^2)^2}.$$

Substituting $t = \sqrt{3}$, we get $-\sqrt{3}/8$ for the acceleration. ■

[6] **11.** A rectangle has two adjacent vertices $(-t, 0)$ and $(t, 0)$ on the x-axis and the other two on the parabola $y = k - x^2$, where $k > 0$. For each k there exists $t > 0$ which maximizes the area of the resulting rectangle.

Find k such that the rectangle of maximum area is a square.

ANSWER:
$k = 3$



SHOW YOUR WORK: From the figure, the area of the rectangle is given by:

$$A = 2t(k - t^2).$$

Now $dA/dt = 2k - 6t^2$ is 0 when $t = \pm\sqrt{k/3}$. Since $dA/dt > 0$ for $t \in (0, \sqrt{k/3})$ and $dA/dt < 0$ for $t \in (\sqrt{k/3}, \sqrt{k})$, $t = \sqrt{k/3}$ gives the maximum area. For the resulting rectangle to be a square, we need

$$2\sqrt{k/3} = 2t = k - t^2 = k - (\sqrt{k/3})^2 = 2k/3.$$

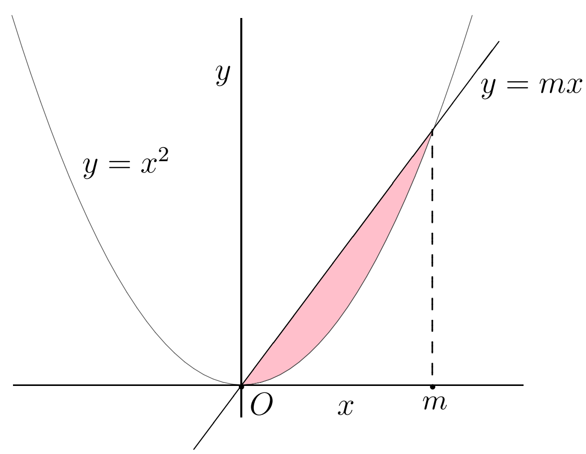
The only solution is $k = 3$ because k is constrained to satisfy $k > 0$. ■

[6] **12.** Find the line $y = mx$ through the origin, with positive slope, which together with the fragment of the parabola

$$y = x^2 \qquad (0 \le x \le m)$$

encloses a region of area $4/3$.

ANSWER:
$y = 2x$



SHOW YOUR WORK: The area below $y = x^2$ from $x = 0$ to m is found to be $\int_0^m x^2\,dx = m^3/3$. Thus the area between $y = m^2x$ and the parabola is $(m^3/2) - (m^3/3) = m^3/6$. For $m^3/6 = 4/3$ we need $m = 2$. ■

[6] **13.** A bacteria-infested swimming pool was chemically treated this morning, and since then, the bacteria count has been decreasing at rate proportional to the count itself.

An hour ago, the count was a third of what it was two hours ago. For safety, the count must be $\leq 1\%$ of what it is now.

When will that be?

ANSWER: In about 4.2 hours

SHOW YOUR WORK: Let $C(t)$ be the bacteria count at time t. It is given that $\dfrac{dC}{dt} = kC$, where k is a constant. This equation can be rewritten as $\dfrac{d}{dt}\ln C = k$. Taking antiderivatives gives $\ln C = kt + b$, where b is a constant. Hence $C = Be^{kt}$, where $B = e^b$. Let t be measured in hours from "now". Then $C(-1) = (1/3)C(-2)$ tells us that $e^k = 1/3$. Clearly, $C(0) = B$. So for the count to be $\leq B/100$ we need $e^{kt} < 100$, which is $(1/3)^t \leq 1/100$. Rearranging we get $3^t \geq 100$, which means $t \geq (\ln 100)/(\ln 3) = \log_3 100 \approx 4.2$. ∎

14. Let $f(x) = 2x^4 - 3x^3 + 5x^2 - 3x + 2$.

[3] (a) Find the line tangent to $y = f(x)$ at $x = 0$.

ANSWER: $y + 3x = 2$

SHOW YOUR WORK: The point on $y = f(x)$ at $x = 0$ is $(0, f(0)) = (0, 2)$. By inspection, $f'(0) = -3$. Hence the tangent at $(0, 2)$ is $y + 3x = 2$. ∎

[3] (b) Show that the line found in (a) intersects $y = f(x)$ only at $x = 0$.

EXPLANATION: The graphs of $y = f(x)$ and $y + 3x = 2$ intersect where $f(x) = 3x + 2$. This gives $2x^4 - 3x^3 + 5x^2 - 3x + 2 = -3x + 2$. Simplifying, we get

$$x^2(2x^2 - 3x + 5) = 0.$$

Since the discriminant of $2x^2 - 3x + 5$ is $(-3)^2 - 4 \cdot 2 \cdot 5 = -31 < 0$, the quadratic has no real solutions. Thus $x = 0$ is the only solution. ∎

[3] (c) Use a linear approximation to estimate $f(0.01)$.

ANSWER: $f(.01) \approx 1.97$

SHOW YOUR WORK: The linear approximation is $f(0) + (.01)f'(0) = 2 - 3(.01)$. ∎

[4] (d) Let a real number a be given as well as the exact value of $f(a)$. Now suppose that a linear approximation is used to estimate $f(a + 0.01)$.

Show that the estimate will be an underestimate whatever the value of a.

EXPLAIN: The intuition is that the estimate will always be an underestimate provided $f'(x)$ is increasing on $(-\infty, \infty)$. Now $f''(x) = 24x^2 - 6x + 10$ has no zeroes because the discriminant is < 0. Hence $f''(x)$ has the same sign for all $x \in (-\infty, \infty)$. Since $f''(0) = 10$, $f''(x) > 0$ for all x. Hence, by the Mean Value Theorem (MVT), $f'(x)$ is an increasing function.

The difference between $f(a + .01)$ and the linear estimate is

$$f(a + .01) - [f(a) + (.01)f'(a)] = [f(a + .01) - f(a)] - (.01)f'(a) = (.01)f'(c) - (.01)f'(a)$$

for some $c \in (a, a + .01)$ by the MVT. Since $c > a$, it is clear that the value $f(a + .01)$ is greater than the estimate.

Those familiar with the Lagrange form of the remainder for a Taylor series may choose to point out that there exists $c \in (a, a + .01)$ such that

$$f(a + .01) = f(a) + (.01)f'(a) + \frac{(.01)^2}{2} f''(c).$$

■