SFU - UBC - UNBC - UVic

Calculus Challenge Exam

June 6, 2002, 14:00 – 17:00

Host: SIMON FRASER UNIVERSITY

Student signature

INSTRUCTIONS

The instructions are distributed separately. Please read them carefully.

Question	Maximum	Score
1	6	
2	6	
3	8	
4	6	
5	6	
6	3	
7	9	
8	10	
9	9	
10	6	
11	6	
12	6	
13	6	
14	13	
Total	100	

1. Compute the following limits.

[3] (a) $\lim_{t \to -2} \dfrac{t^2 - t - 6}{t^2 + 5t + 6}$

ANSWER:

JUSTIFY YOUR ANSWER

[3] (b) $\lim_{x \to 0^+} \left[\dfrac{1}{x} \left(3e^{-1/x} + 5e^x \right) \sin x \right]$

ANSWER:

JUSTIFY YOUR ANSWER

[4] **2.** (a) Find the asymptotes of $y = \left(\frac{x}{x-1}\right)^2$ and justify your answer.

ANSWER:

EXPLANATION

[2] (b) Where does the curve $y = \left(\frac{x}{x-1}\right)^2$ cross its horizontal aymptote?

ANSWER:

EXPLANATION

[4] **3.** (a) Find $\frac{dv}{du}$ when $v = \sqrt{\frac{\tan u}{1 + \tan u}}$.

ANSWER:

$\frac{dv}{du} =$

SHOW YOUR WORK

[4] (b) Let a be a constant and $f(x) = \sin(ax)$. Find the 97-th derivative, $f^{(97)}(x)$, of the function $f(x)$.

ANSWER:

SHOW YOUR WORK

[3] **4.** (a) Find the general antiderivative of $\left(9 - 4x^2\right)^{-1/2}$.

ANSWER:

SHOW YOUR WORK

[3] (b) It is given that

$$f'(x) = 2^x + x^2 \quad \text{and} \quad f(0) = 0.$$

Find $f(x)$.

ANSWER:

SHOW YOUR WORK

[6] **5.** Use the definition of derivative (and <u>not</u> the product rule) to show that, if $f(x)$ is differentiable at $x = c$ and $g(x) = xf(x)$, then $g'(c)$ exists and $g'(c) = f(c) + cf'(c)$.

ANSWER

[3] **6.** For what value of k is the function

$$h(x) = \begin{cases} 2x + 3 & \text{if } x \leq 1 \\ k - 1 & \text{if } x > 1 \end{cases}$$

continuous?

ANSWER:

JUSTIFY YOUR ANSWER

[4] **7.** (a) Express $\frac{dy}{dx}$ as a function of x, when $y = \left(\frac{x^7 \cos x}{7^x \sqrt{1+x^2}}\right)$.

ANSWER:

$\frac{dy}{dx} =$

SHOW YOUR WORK

[5] (b) Express $\frac{dy}{dx}$ as a function of x, when $y = x^{\ln x}$.

ANSWER:

$\frac{dy}{dx} =$

SHOW YOUR WORK

8. A curve has the equation $\sin(x+y) = xe^y$.

[2] (a) Show that $(0, \pi)$ is on the curve.

ANSWER

[4] (b) Find the equation of the line tangent to the curve at $(0, \pi)$.

ANSWER:

SHOW YOUR WORK

[4] (c) A point moves along the curve so that at $(0, \pi)$ its x-coordinate is increasing at a rate of 3 units/sec. How fast is its y-coordinate changing at $(0, \pi)$?

ANSWER:

SHOW YOUR WORK

9. Let $f(x) = e^{x-2} + x^3 - 2$.

[3] (a) Use the derivative of f to explain why the equation $f(x) = 0$ has at most one solution.

EXPLANATION

[3] (b) Explain why $f(x) = 0$ has a solution in the interval $(1, 2)$.

EXPLANATION

[3] (c) Newton's method with an initial estimate of 2 is used to find an approximate value for the solution of $f(x) = 0$.

What is the next estimate?

ANSWER:

SHOW YOUR WORK

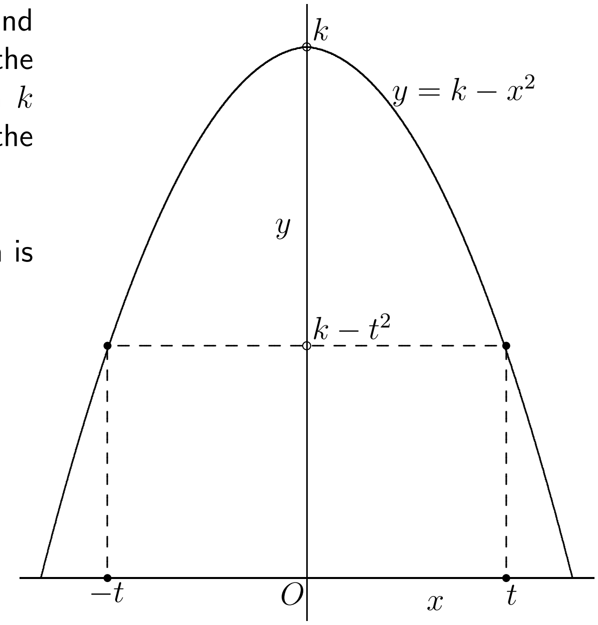
[6] **10.** A particle moves along the x-axis with velocity $\dfrac{1}{1+t^2}$ at time t. If it passes the point $\pi/6$ at time $t = 1$, what is its acceleration when it passes the point $\pi/4$?

ANSWER:

SHOW YOUR WORK

[6] **11.** A rectangle has two adjacent vertices $(-t, 0)$ and $(t, 0)$ on the x-axis and the other two on the parabola $y = k - x^2$, where $k > 0$. For each k there exists $t > 0$ which maximizes the area of the resulting rectangle.

Find k such that the rectangle of maximum area is a square.



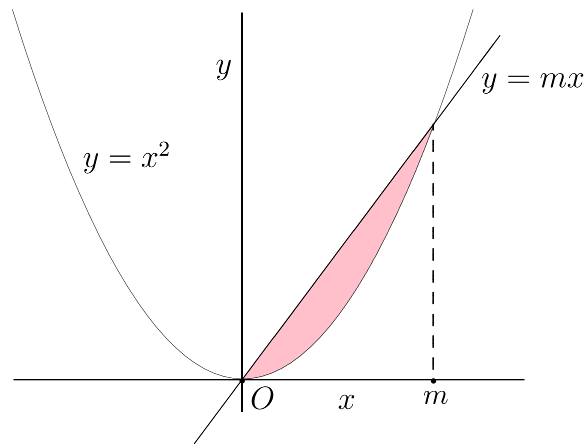
ANSWER:

SHOW YOUR WORK

[6] **12.** Find the line $y = mx$ through the origin, with positive slope, which together with the fragment of the parabola

$$y = x^2 \qquad (0 \leq x \leq m)$$

encloses a region of area $4/3$.



ANSWER:

SHOW YOUR WORK

[6] **13.** A bacteria-infested swimming pool was chemically treated this morning, and since then, the bacteria count has been decreasing at rate proportional to the count itself.

An hour ago, the count was a third of what it was two hours ago. For safety, the count must be $\leq 1\%$ of what it is now.

When will that be?

ANSWER:

SHOW YOUR WORK

14. Let $f(x) = 2x^4 - 3x^3 + 5x^2 - 3x + 2$.

[3] (a) Find the line tangent to $y = f(x)$ at $x = 0$.

ANSWER:

SHOW YOUR WORK

[3] (b) Show that the line found in (a) intersects $y = f(x)$ only at $x = 0$.

EXPLANATION

[3] (c) Use a linear approximation to estimate $f(0.01)$.

ANSWER:

SHOW YOUR WORK

[4] (d) Let a real number a be given as well as the exact value of $f(a)$. Now suppose that a linear approximation is used to estimate $f(a + 0.01)$.

Show that the estimate will be an underestimate whatever the value of a.

EXPLAIN