Lior Silberman's Math 412: Problem set 10, due 5/4/2023

P1. Recall that a *projection* is a linear map E such that $E^2 = E$. For each n construct a projection $E_n \colon \mathbb{R}^2 \to \mathbb{R}^2$ of norm at least n ($\mathbb{R}^n$ is equipped with the Euclidean norm unless specified otherwise). Prove for yourself that the norm of an *orthogonal* projection is 1.

Difference and differential equations

P2. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Let $\underline{v}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) Find S invertible and D diagonal such that $A = S^{-1}DS$.

– Prove for yourself the formula $A^k = S^{-1}D^kS$.

(b) Find a formula for $\underline{v}_k = A^k\underline{v}_0$, and show that $\frac{\underline{v}_k}{\|\underline{v}_k\|}$ converges for any norm on $\mathbb{R}^2$.

RMK You have found a formula for Fibbonacci numbers (why?), and have shown that the real number $\frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n$ is exponentially close to being an integer.

RMK This idea can solve any *difference equation*, and also *differential equations*.

1. We will analyze the differential equation $u'' = -u$ with initial data $u(0) = u_0$, $u'(0) = u_1$.

(a) Let $\underline{v}(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$. Show that u is a solution to the equation iff $\underline{v}$ solves

$$\underline{v}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underline{v}(t)\,.$$

(b) Let $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Find a formula for W^n and express $\exp(Wt) = \sum_{k=0}^{\infty} \frac{W^k t^k}{k!}$ as a matrix whose entries are standard power series.

(c) Sum the series and show that $u(t) = u_0 \cos(t) + u_1 \sin(t)$.

(d) Find a matrix S such that $W = S \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} S^{-1}$. Evaluate $\exp(Wt)$ again, this time using $\exp(Wt) = S\left(\exp\begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix}\right) S^{-1}$.

DEF The *companion matrix* associated to the polynomial $p(x) = x^n - \sum_{i=0}^{n-1} a_i x^i$ is

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}.$$

2. A sequence $\{x_k\}_{k=0}^{\infty}$ is said to satisfy a *linear recurrence relation* (or a *difference equation*) if for each k,

$$x_{k+n} = \sum_{i=0}^{n-1} a_i x_{k+i}\,.$$

(a) Define vectors $\underline{v}^{(k)} = (x_{k-n+1}, x_{k-n+2}, \ldots, x_k)$. Show that $\underline{v}^{(k+1)} = C\underline{v}^{(k)}$ where C is the companion matrix above.

(b) Find x_{100} if $x_0 = 1$, $x_1 = 2$, $x_2 = 3$ and $x_n = x_{n-1} + x_{n-2} - x_{n-3}$.

hint: Find the Jordan canonical form of $\begin{pmatrix} & 1 & \\ & & 1 \\ -1 & 1 & 1 \end{pmatrix}$.

3. Let C be the companion matrix associated with the polynomial $p(x) = x^n - \sum_{k=0}^{n-1} a_k x^k$.
   (a) Show that $p(x)$ is the characteristic polynomial of C.
   (b) Show that $p(x)$ is also the minimal polynomial.
   – For parts (c),(d) fix a non-zero root λ of $p(x)$.
   (c) Find (with proof) an eigenvector with eigenvalue λ.
   (**d) Let g be a polynomial, and let $\underline{v}$ be the vector with entries $v_k = \lambda^k g(k)$ for $0 \le k \le n-1$. Show that, if the degree of g is small enough (depending on p, λ), then $((C-\lambda)\,\underline{v})_k = \lambda\,(g(k+1) - g(k))\,\lambda^k$ and (the hard part) that

$$((C-\lambda)\underline{v})_{n-1} = \lambda\,(g(n) - g(n-1))\,\lambda^{n-1}\,.$$

   (**e) Find the Jordan canonical form of C.

4. Consider now *differential* equation $\frac{\mathrm{d}}{\mathrm{d}t}\underline{v} = B\underline{v}$ where B is at in PS8 problem 1.
   (a) Find matrices S, D so that D is in Jordan form, and such that $B = SDS^{-1}$.
   (b) Find $\exp(tD)$ as in 1(b) by computing a formula for D^n and summing the series.
   (c) Find the solution such that $\underline{v}(0) = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}^{\mathrm{t}}$.

Power series

5. Let $A = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$ with $z \in \mathbb{C}$.
   (a) Find a simple formula for the entries of A^n.
   (b) Use your formula to decide the set of z for which $\sum_{n=0}^{\infty} A^n$ converge, and give a formula for the sum.
   (c) Show that the sum is $(\mathrm{Id} - A)^{-1}$ when the series converges.

6. For any matrix $A \in M_n(\mathbb{C})$ show that $\sum_{n=0}^{\infty} z^n A^n$ converges for $|z| < \frac{1}{\rho(A)}$.
   Hint: see PS9 problem 3.

Supplementary problems

A. Consider the map $\mathrm{Tr}\colon M_n(F) \to F$.
   (a) Show that this is a continuous map.
   (b) Find the norm of this map when $M_n(F)$ is equipped with the $L^1 \to L^1$ operator norm (see PS9 Problem 2(a)).
   (c) Find the norm of this map when $M_n(F)$ is equipped with the Hilbert–Schmidt norm (see PS9 Problem A).
   (*d) Find the norm of this map when $M_n(F)$ is equipped with the $L^p \to L^p$ operator norm. Find the matrices A with operator norm 1 and trace maximal in absolute value.

B. Call $T \in \mathrm{End}_F(V)$ *bounded below* if there is $K > 0$ such that $\|T\underline{v}\| \ge K\,\|\underline{v}\|$ for all $\underline{v} \in V$.
   (a) Let T be boudned below. Show that T is invertible, and that T^{-1} is a bounded operator.
   (*b) Suppose that V is finite-dimensional. Show that every invertible map is bounded below.

C. (The supremum norm and the Weierestrass M-test) Let V be a complete normed space.
   DEF For a set X call $f\colon X \to V$ *bounded* if there is $M > 0$ such that $\|f(x)\|_V \le M$ for all $x \in X$ in which case we write $\|f\|_\infty = \sup_{x \in X} \|f(x)\|_V$ (equivalently, f is bounded if $x \mapsto \|f(x)\|_V$ is in $\ell^\infty(X)$).
   (a) Show that $\ell^\infty(X; V)$ is a vector space (this doesn't use completeness of V).
   (b) Show that $\ell^\infty(X; V)$ is complete.
   DEF Now suppose that X is a metric space (or, more generally, a topological space). Let $C(X; V)$ denote the space of *continuous* functions $X \to V$ and let $C_{\mathrm{b}}(X; V) = C(X; V) \cap \ell^\infty(X; V)$ be the space of *bounded* continuous functions, the latter equipped with the ℓ^∞-norm.
   (c) Show that $C_{\mathrm{b}}(X; V)$ is a closed subspace of $\ell^\infty(X; V)$. Conclude that it is complete.
   COR Deduce Weirestrass's M-test: $f_n\colon X \to V$ are continuous and satisfy $\|f_n\|_\infty \le M_n$ with $\sum_n M_n < \infty$ then $\sum_n f_n$ converges to a continuous function of norm at most by $\sum_n M_n$.