Lior Silberman's Math 223: Problem Set 9 (due 21/3/2022)

Calculation

M1. Find the eigenvalues and a basis for each eigenspace for the matrices $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

M2. (Complex numbers)
   (a) Compute: $(5+3i)+(6+7i)$, $(5+3i)\cdot(6+7i)$, $\frac{5+3i}{6+7i}$ (hint: $\frac{1}{6+7i} = \frac{6-7i}{(6+7i)(6-7i)}$).
   (b) Let $w = a+bi$ be a non-zero complex number. Show that there are two complex solutions to the equation $z^2 = w$. (Hint: write $z = x+yi$ and get a system of two equations in the unknowns x,y).
   (c) Let $a,b,c \in \mathbb{C}$ with $a \neq 0$. Show that the polynomial $az^2+bz+c \in \mathbb{C}[z]$ factors as a product of linear polynomials. (Hint: use the quadratic formula)

Three determinants

Hint for 1,2,3: if you aren't sure try what happens with small matrices ($2\times 2, 3\times 3, 4\times 4, 5\times 5$) before tackling the general case.

1. Fix numbers a,b and let T_n be the matrix with entries t_{ij} so that for all i, $t_{ii} = a$, $t_{i,(i-1)} = t_{i,(i+1)} = b$ and $t_{ij} = 0$ otherwise.
   (a) For $n \geq 1$ use row and column expansion to show that $\det T_{n+2} = a(\det T_{n+1}) - b^2(\det T_n)$.
   (b) Using the method of problem 5 below solve the recursion in the case $a = 5$, $b = 2$ and find a closed-form expression for $\det T_n$ in this case.

2. Let $H_n(d_1,\cdots,d_n)$ be the matrix $J_n + \mathrm{diag}(d_1,\ldots,d_n)$ where J_n is the all-ones matrix and let $h_n(d_1,\cdots,d_n) = \det[H_n(d_1,\cdots,d_n)]$.
   (a) Show that $h_n(0,d_2,\ldots,d_n) = \prod_{j=2}^n d_j$. (Hint: subtract the second row from the first)
   (b) Suppose that $n \geq 2$. Show that $h_n(d_1,d_2,\ldots,d_n) = d_1 h_{n-1}(d_2,\ldots,d_n) + d_2 h_{n-1}(0,d_3,\ldots,d_n)$.
   (c) Suppose that all the $d_i \neq 0$ and that $n \geq 2$. Show that $\frac{h_n(d_1,\ldots,d_n)}{\prod_{j=1}^n d_j} = \frac{h_{n-1}(d_2,\ldots,d_n)}{\prod_{j=2}^n d_j} + \frac{1}{d_1}$.
   (d) Show that $\frac{h_2(d_1,d_2)}{d_1 d_2} = \frac{1}{d_1} + \frac{1}{d_2} + 1$, and thus that $\frac{h_n(d_1,\ldots,d_n)}{\prod_{j=1}^n d_j} = \sum_{j=1}^n \frac{1}{d_j} + 1$.

   CONCLUSION $h_n(d_1,\ldots,d_n) = \left(\sum_{j=1}^n \frac{1}{d_j} + 1\right)\left(\prod_{j=1}^n d_j\right)$.

3. (The "Vandermonde determinant") Let x_i be variables and let $V_n(x_1,\ldots,x_n)$ be the $n\times n$ matrix with entries $v_{ij} = x_i^{j-1}$. We show that $\det V_n = \prod_{i=2}^n \prod_{j=1}^{i-1}(x_i - x_j)$.
   (a) Show that $\det V_n$ is a polynomial in $x_1,\ldots,x_n$ of total degree $0+1+2+3+\cdots+(n-1) = \frac{n(n-1)}{2}$.
   (b) Show that $\det V_n$ vanishes whenver $x_i = x_j$ (which leads you to suspect that $x_i - x_j$ divides the polynomial).

   RMK Note that $\prod_{i=2}^n \prod_{j=1}^{i-1}(x_i - x_j)$ is a polynomial of total degree $\frac{n(n-1)}{2}$. It follows from (a) and the theory of polynomial rings over integral domains that $\prod_{i=2}^n \prod_{j=1}^{i-1}(x_i - x_j)$ actually does divide the determinant, and comparing degrees of the two it follows that the quotient has degree zero, that is that for some constant $c_n \in \mathbb{Z}$, $\det V_n = c_n \prod_{i=2}^n \prod_{j=1}^{i}(x_i - x_j)$. In the rest of the problem we prove this formula without this theory.

   SUPP Examining the coefficient of $x_1^0 x_2^1 x_3^2 \cdots x_n^{n-1}$ show that $c_n = 1$.

(c) Let $V_{n+1}(x_1,\ldots,x_{n+1})$ be the matrix described above, and let W_{n+1} be the matrix obtained by
  (i) Subtracting the first row from each row; and then
  (ii) For j descending from $n+1$ to 2, subtracting from the jth column a multiple of the $(j-1)$st so as to make the top entry in the column zero.

  Let $\left(w_{ij}\right)_{i,j=1}^{n+1}$ be the entries of W_{n+1}. Show that $w_{11}=1$ that $w_{1j}=w_{i1}=0$ if $i,j\neq 1$ and that $w_{ij}=(x_i-x_1)v_{i,j-1}$ if $i,j\geq 2$.

(d) Show that $\det V_{n+1}=\left[\prod_{i=2}^{n+1}(x_i-x_1)\right]\cdot[\det V_n(x_2,\ldots,x_{n+1})]$.

(e) Check that $\det V_1=1$ and prove the main claim by induction.

SUPP (Polynomial interpolation) Let $\{(x_i,y_i)\}_{i=1}^{k}\subset\mathbb{R}^2$ be points in the plane with distinct x_i. Show that there exists a unique polynomial $p\in\mathbb{R}[x]^{<k}$ such that $p(x_i)=y_i$.

Linear recurrences

4. Let $T\in\operatorname{End}(V)$ and let $\underline{v}\in V$ satisfy $T\underline{v}=\lambda\underline{v}$.
  (a) Show that $T^n\underline{v}=\lambda^n\underline{v}$ for all $n\geq 0$.
  (b) Suppose that T is invertible and $\underline{v}\neq 0$. Show that $\lambda\neq 0$ and that $T^{-n}\underline{v}=\lambda^{-n}\underline{v}$.
  (c) Let $p\in\mathbb{R}[x]$ be a polynomial of degree n. Show that $p(T)\underline{v}=p(\lambda)\underline{v}$, where $p(T)$ is the linear map defined in the supplement to PS6.

5. A sequence $\underline{F}\in\mathbb{C}^{\mathbb{N}}$ satisfies a *recurrence relation of degree* k if we have coefficients $c_0,\ldots,c_{k-1}$ with $c_0\neq 0$ such that $F_{n+k}=\sum_{i=0}^{k-1}c_iF_{n+i}$ for all $n\geq 0$. In that case let $p(x)=x^k-\sum_{i=0}^{k-1}c_ix^i$ be the *characteristic polynomial* of the recursion relation.
  (a) Explain why we assume $c_0\neq 0$.
  (b) Show that $\underline{F}$ satisfies the recurrence relation iff $p(L)\underline{F}=\underline{0}$, where $L\in\operatorname{End}(\mathbb{R}^\infty)$ is the left shift.

  RMK In other words, we show that the set of solutions to the recurrence relation is the kernel of a linear map.

  (c) Show that any solution is determined by its initial values $(F_0,F_1,\ldots,F_{k-1})$. Conclude that $\operatorname{Ker}(p(L))$ is k-dimensional.
  (d) Suppose that r is a root of $p(x)$. Show that the sequence $(r^n)_{n\geq 0}$ is in $\operatorname{Ker}(p(L))$ (i.e. that it satisfies the recursion relation) and that it is non-zero.

  FACT A set of (non-zero) eigenvectors corresponding to distinct eigenvalues is linearly independent.

  ASSUME for the rest of the problem that $p(x)$ has k distinct roots $\{r_i\}_{i=1}^{k}$.

  (e) Find a basis for $\operatorname{Ker}(p(L))$.
  (f) Let $(F_0,F_1,\ldots,F_{k-1})$ be any numbers. Show that the system of k equations $\sum_{i=0}^{k-1}A_ir_i^j=F_j$ $(1\leq j\leq k)$ in the unknowns A_i has a unique solution. (Hint: problem 3)

  SUPP Show (without using the assumption) for any recurrence relation of degree k, any initial k-tuple of values extends to a unique solution of the recurrence relation.

Extra Credit: Practice with Incidence geometry

An *incidence structure* is a triple $(P,L,\in)$ where P is a set (its elements are called *points*), L is a set (its elements are called "lines"), and $\in$ is an (arbitrary) relation between the sets P,L called "incidence". We interpret the situation $p\in\ell$ as "the point p lies on the line ℓ" (is incident to it) and $p\notin\ell$ to be the reverse sitaution. We always assume that P,L are finite. The incidence structure is a *linear space* if we add the axiom "for any two distinct points $p\neq p'$ there is a unique line ℓ such that $p\in\ell$ and $p'\in\ell$.

Our goal is to prove

THEOREM (De Bruin–Erdős). *Let $(P,L,\in)$ be an linear space for which not all points are on the same line. Then there are at least as many lines as points.*

*C1. Let $(P, L, \in)$ be a linear space.

(a) Suppose that for some point p there is only one line containing p. Show that this line contains all points.

DEF Let $T\colon \mathbb{R}^P \to \mathbb{R}^L$, $S\colon \mathbb{R}^L \to \mathbb{R}^P$ be the maps $(Tf)(\ell) = \sum_{p\in\ell} f(p)$ (sum over points on ℓ) and $(Sg)(p) = \sum_{p\in\ell} g(\ell)$ (sum over lines containing p).

(b) Show that T, S are linear.

(c) Suppose that $P = \{p_i\}_{i=1}^n$ is finite. Show that the matrix of ST in the "standard basis" of $\mathbb{R}^P$ (the ith basis vector is the function which is 1 at p_i, zero elsewhere) is $J_n + \operatorname{diag}(d_1 - 1, \ldots, d_n - 1)$ where J_n is the all-ones matrix and d_i is the number of lines through p_i.

(d) Suppose that not all points are on the same line. Show that $\det(ST) > 0$.

(e) Prove the Theorem.

C2. Suppose that we add the axiom "every two distinct lines intersect at exactly one point".

(a) Show that in this case exchanging the role of points and lines (and the adjusting the relation appropriately) gives a new incidence structure (the "dual one") which also satisfies both the linear space axiom and the axiom we just introduced.

(b) Conclude that with the extra axiom there only three possibilities: (1) there is exactly one line and it contains all the points; (2) there is exactly one point and it lies on all lines; (3) there are as many lines as points

Supplementary problem: Quadratic extensions in general

A (Constructing quadratic fields) Let F be a field, $d \in F$ such that $x^2 = d$ has no solutions in F.

(a) Show that the set of matrices $E = \left\{ \begin{pmatrix} a & b \\ db & a \end{pmatrix} \mid a, b \in F \right\}$ is a two-dimensional F-subspace of $M_2(F)$ with basis $1, \varepsilon$, where $\varepsilon = \begin{pmatrix} & 1 \\ d & \end{pmatrix}$ satisfies $\varepsilon^2 = d$.

(b) Show that E is also closed under matrix multiplication and that $xy = yx$ for all $x, y \in E$.

(c) Show that the unique F-linear map $\sigma\colon E \to E$ given by $\sigma(I_2) = I_2$, $\sigma(\varepsilon) = -\varepsilon$, satisfies $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in E$.

(d) Show that the norm $Nz = z\sigma(z)$ satisfies $Nz \in F$ for all $z \in E$, $Nz \neq 0$ iff $z \neq 0$, $N(zw) = NzNw$.

(e) Conclud that E is a field.

B. (Uniqueness) Let E' be a field containing F which is two-dimensional over F.

(a) Suppose E' is spanned over F by elements $1, \varepsilon$ with $\varepsilon^2 = d$. Let $z = a + b\sqrt{d} \in E'$ be any element and let $M_z\colon E' \to E'$ be the map of multiplication by z. Show that M_z is F-linear and that its matrix in the basis $\{1, \varepsilon\}$ is $\begin{pmatrix} a & b \\ db & a \end{pmatrix}$.

(b) Show that E always has a basis of the form $\{1, \delta\}$ with $\delta \notin F$. Show that if $\operatorname{char} F \neq 2$ there is $\varepsilon = a + b\delta$ such that $\varepsilon^2 \in F$.

(c) Show that $E = F(\sqrt{d})$ and $E' = F(\sqrt{d'})$ are isomorphic as fields iff $\frac{d}{d'}$ is a square in F.