Lior Silberman's Math 223: Problem Set 7 (due 7/3/2022)

Practice problems (recommended, but do not submit)

Section 3.2, Problems 1-8.
Section 3.3, Problems 1-8, 10.
Section 3.4, Problems 1-8

Calculation

M1. Solve the following systems of linear equations using Gaussian elimination. Follow the following procedure: (1) Write the augmented matrix (2) note every step of your calculation (e.g. "add 2x the third row to the first") (3) Once you reach reduced row echelon form read off the solution. (4) Circle every pivot in the original matrix from step 1. If you swapped rows this need not be at the location you used it.

(a) $\begin{cases} 2x+3y & =5 \\ 3x+5y & =7 \end{cases}$ (b) $\begin{cases} 2x+3y+5z & =2 \\ x+2y-z & =1 \end{cases}$ (c) $\begin{cases} 2x+3y+4z & =7 \\ x-2y+z & =5 \\ 7y+2z & =3 \end{cases}$

M2. For each of the following matrices use Gaussian elimination to decide if it is invertible, and to construct an inverse if the matrix is invertible.

(a) $A=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ (b) $B=\begin{pmatrix} 8 & 3 \\ 16 & 6 \end{pmatrix}$ (c) $C=\begin{pmatrix} 1 & -2 & 0 \\ 0 & 7 & 3 \\ 1 & 7 & 2 \end{pmatrix}$.

M3. Let $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $C=\begin{pmatrix} 3 & 5 & 7 \\ 4 & 5 & 6 \\ 7 & 1 & 9 \end{pmatrix}$, $D=BC$.

(a) Evaluate the determinant of the above matrices using minor expansion. Check that $\det(BC)=\det B\cdot\det C$.
(b) Evaluate the same determinants using Gaussian elimination.

Matrices and powers

1. (powers of diagonal and diagonable matrices)
PRAC Let $A=\operatorname{diag}(a_1,\ldots,a_n)$, $B=\operatorname{diag}(b_1,\ldots,b_n)\in M_n(\mathbb{R})$. Show that $A+B=\operatorname{diag}(a_1+b_1,\ldots,a_n+b_n)$ and that $AB=\operatorname{diag}(a_1b_1,\ldots,a_nb_n)$.
(a) Let $A=\operatorname{diag}(a_1,\ldots,a_n)\in M_n(\mathbb{R})$ be diagonal, and let $P(k)$ be the statement "$A^k=\operatorname{diag}\left(a_1^k,\ldots,a_n^k\right)$". Show that $P(0)$ holds and that if $P(k)$ is true then so is $P(k+1)$. From the principle of mathematical induction it follows that $P(k)$ is true for all k.
(b) Use the identity $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}=\begin{pmatrix} 26 & -18 \\ 36 & -25 \end{pmatrix}$ to calculate $\begin{pmatrix} 26 & -18 \\ 36 & -25 \end{pmatrix}^{1000}$.

2. Let F_n be the sequence defined by $F_0=a$, $F_1=b$ and $F_{n+2}=F_{n+1}+F_n$ for all $n\geq 0$.
(a) Show that $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}=\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix}$ holds for all n.
(b) Show that $\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n\begin{pmatrix} a \\ b \end{pmatrix}$ (hint: use the same scheme as in 3(a)).

RMK $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is similar to a diagonal matrix; the idea of 3(b) will give a simple formula for F_n.

Permutations

3. Let X,Y be finite sets of the same size n. Show that the following are equivalent for a function $\sigma\colon X\to Y$:
(1) σ is bijective; (2) σ is injective; (3) σ is surjective.
Hint: Induction on n.
DEF Recall that $[n]=\{1,2,\dots,n\}$. A bijection $[n]\to[n]$ is called a *permutation*. Write S_n for the set of permutations of $[n]$.

4. A *permutation matrix* is an $n\times n$ matrix P having all entries zero except that in every row and column there is exactly one 1. Examples include I_n, $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$. Show that $\det P\in\{\pm1\}$ for every permutation matrix.
RMK If σ is a permutation the corresponding permutation matrix has $P_{ij}=\begin{cases}1 & j=\sigma(i)\\ 0 & j\neq\sigma(i)\end{cases}$.

Extra credit: the Fifteen puzzle

C1. The "fifteen puzzle" is played on a $n\times n$ grid. The puzzle consists of n^2-1 sliders, labelled with the numbers between 1 through n^2-1, placed on distinct grid points, leaving one grid point empty. We will call such a placement a *configuation* of the puzzle. A *legal move* consists of sliding one of the sliders vertically or horizontally into the empty position. For the purposes of a mathematical description we will replace the empty position with an additional slider marked "n^2", so that a configuation consists of a matrix $C\in M_n(\mathbb{R})$ with the entries being $1,2,3,\cdots,n^2$ in some order, and legal moves consists of exchanging the token marked "n^2" with one of its neighbours.
DEF To go through the grid points in "natural order" means to go through the first row in order left-to-right, then the second row left-to-right and so on. We say a grid position occurs "later" than another if it will be checked later when going through the grid in order. Define the *number of crossings* of a configuation to be the number of pairs of grid points such that the number written in the later position of the two is smaller than the number written in the earlier one. Now define the *parity* $\varepsilon(C)$ of a configuration to be $+1$ if there is an even number of crossings, -1 if there is an odd one. Define the *total parity* to be the number $\delta(C)=\varepsilon(C)\times(-1)^{i+j}$ where (i,j) are the coordinates of the position marked n^2.
EXAMPLE $(n=3)$ Let $C=\begin{bmatrix}2&1&5\\9&8&3\\4&6&7\end{bmatrix}$. Then the legal moves are to exchange the 9 with the $2,8$ or 4, the crossings are (in terms of the numbers written in the grid points, not in term of positions) $2\to1$, $5\to3$, $5\to4$, $9\to8$, $9\to3$, $9\to4$, $9\to6$, $9\to7$, $8\to3$, $8\to4$, $8\to6$, $8\to7$, the parity is $(-1)^{12}=1$ and the total parity is $(-1)^{10}(-1)^{2+1}=-1$ since the 9 is in position $2,1$.
(**a) Let C,C' be two positions connected by a single legal move. Show that $\varepsilon(C)=-\varepsilon(C')$ and that $\delta(C)=\delta(C')$.
(b) Let C,C' be two positions such that we can go from C to C' by $m\geq0$ legal moves. Show that $\delta(C)=\delta(C')$.
(c) (Negative solution to the Fifteen Puzzle) Show that there is no sequence of legal moves that starts in the configuation $\begin{bmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\\13&15&14&E\end{bmatrix}$ and ends in the configuation $\begin{bmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\\13&14&15&E\end{bmatrix}$. Here we denoted the empty position E rather than 16.

C2. (The sign representation) For a permutation $\sigma \in S_n$, a *crossing* is a pair of numbers $i < j$ such that $\sigma(j) < \sigma(i)$. So for example the *identity* permutation $\sigma(i) = i$ has no crossings at all, whereas for the *reveresal permutation* $\sigma(i) = n+1-i$ all pairs cross, so there are $\frac{n(n+1)}{2}$ of them. Write $t(\sigma)$ for the number of crossings and $(-1)^\sigma = (-1)^{t(\sigma)}$.

(a) A *swap* in a permutation is exchanging the values of $\sigma(i)$ and $\sigma(j)$. Show that if τ is obtained from σ by a swap then $t(\sigma)$ and $t(\tau)$ have different parities, so that $(-1)^\tau = -(-1)^\sigma$.

(b) Conclude that if we can obtain σ from the identity permutation by doing k swaps then $(-1)^\sigma = (-1)^k$.