Math 535: Real Groups
Lecture Notes

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These are rough notes for the Spring 2018 course. Solutions to problem sets were posted on an internal website.

Contents

Introduction

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0.1. Administrivia

- Problem sets will be posted on the course website.
  - To the extent I have time, solutions may be posted on Connect.
- Textbooks
  - Warner, Lee
  - Bröcker–tom Dieck, *Representations of Compact Lie Groups*, GTM-98
  - Knapp, *Lie groups beyond an introduction*
  - Knapp, *Representation Theory of Semisimple Groups*
- No exams.

CHAPTER 1

Basics: Locally compact groups and their representations

REMARK 1. On foundations.

1.1. Topological groups

DEFINITION 2. A *topological group* is a group object in the category of Hausdorff topological spaces. A *homomorphism* of topological groups is a continuous group homomorphism. An *action* of the topological group G on the topological space X is a group action $\cdot\colon G \times X \to X$ which is continuous for the product topology on $G \times X$.

Note that the regular action of G on itself is a continuous action by homeomorphisms.

EXAMPLE 3. $\mathbb{R}$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{Q})$, $\mathbb{Q}_p$, C_2^X (X arbitrary!), etc.

LEMMA 4. *Suffices to assume T_1, that is that $\{e\} \subset G$ is closed.*

PROOF. By the invariance of the topology if $\{e\}$ is closed so is every point, and it is enough to separate e from g for every $g \neq e$. Since the group is T_1, the set $G \setminus \{g\}$ is open. By continuity of the map $(x,y) \mapsto xy^{-1}$ at the identity there is a neighbourhood $(e,e) \in U \times V \subset G \times G$ such that $xy^{-1} \neq g$ for al $(x,y) \in U \times V$. Then $W = U \cap V$ works. □

LEMMA 5. *Let $H \subset G$ be a subgroup. Then the quotient topology on G/H is Hausdorff iff H is closed.*

PROOF. Let $q\colon G \to G/H$ be the quotient map. If G/H is Hausdorff it is T_1 so $H = q^{-1}(H)$ is closed. Conversely, if H is closed by invariance it is enough to separate $H, gH \in G/H$. For that let $W \subset G$ be a neighbourhood of the identity such that $W^{-1}W \cap gH = \emptyset$. It then follows that $W^{-1}WH \cap gH = \emptyset$ as well. It follows that the open sets WH and WgH are disjoint, and hence that their (open) images in G/H are disjoint. □

1.2. Representation Theory

1.2.1. Continuous representations.

DEFINITION 6. A *representation* π of the topological group G on the TVS V_π is a continuous action by linear maps. A *unitary* representation is a represetation on a Hilbert space V_π by unitary maps.

DEFINITION 7. Let (π, V) and (σ, W) be representations of G. An *intertwining operator* (or *G-homomorphism*) between them is a continuous map $f\colon V \to W$ such that

$$\forall g \in G\colon \sigma(g) \circ f = f \circ \pi(g)\,.$$

We will write $\mathrm{Hom}_G(V,W)$ for the set of G-homomorphisms, $\mathrm{Rep}(G)$ for the category of representations and G-homomorphisms.

LEMMA 8. *Let $(\pi, V) \in \mathrm{Rep}(G)$. If $W \subset V$ is G-invariant then so is its closure $\bar{W}$.*

DEFINITION 9. Call (π, V) (topologically) *irreducible* if its only closed G-invariant subspaces are the obvious ones.

EXAMPLE 10. Fix a group G.

(1) The *trivial representation* is the unique representation with $V = \{\underline{0}\}$.
(2) For any reasonable function space, including $C(G)$, $L^2(G)$ (if G is locally compact and unimodular)

1.2.2. Constructions.

LEMMA-DEFINITION 11. *Let (π, V) and (σ, W) be representations of G.*

(1) *For $g \in G$ set $\check{\pi}(g) =^t \pi(g)^{-1}$. Then $\check{\pi}$defines a representation of G on the continuous dual V'.*
(2) *Endowing $V \oplus W$ with the product topology, setting $(\pi \oplus \sigma)(g) = \pi(g) \oplus \sigma(g)$ defines a representation.*
(3) *Suppose $U \subset V$ is a G-invariant closed subspace. Then setting $\bar{\pi}(g)(\underline{v} + U) = \pi(g)\underline{v} + U$ defines a continuous representation of G on V/U.*

PROOF. Exercise. □

LEMMA-DEFINITION 12 (Naive tensor product). *Let $(\pi, V), (\sigma, W)$ be representations of G, H respectively. Then $G \times H$ acts on the algebraic tensor product $V \otimes W$ by $(\pi \otimes \sigma)(g, h) \stackrel{def}{=} \pi(g) \otimes \sigma(h)$.*

REMARK 13. When V, W are finite-dimensional so is $V \otimes W$ and there is no problem with the topology.

1.2.3. Matrix coefficients.

DEFINITION 14. Let (π, V) be a representation of G. A *matrix coefficient* of V is any function

$$\Phi_{\underline{v}, \underline{v}'}(g) = \langle \pi(g)\underline{v}, \underline{v}' \rangle$$

where $\underline{v} \in V$, $\underline{v}' \in V'$.

REMARK 15. It is always the case that $\Phi_{\underline{v}, \underline{v}'} \in C(G)$. Further analytic properties of the matrix coefficients (smoothness and decay) are very important.

LEMMA 16. *The map $(\underline{v}, \underline{v}') \mapsto \Phi_{\underline{v}, \underline{v}'}$ is bilinear; the resulting map $V \otimes \check{V} \to C(G)$ is an intertwining operator where $G \times G$ acts on $C(G)$ the right by $((g_1, g_2) \cdot f)(x) = f\left(g_2^{-1} x g_1\right)$.*

PROOF. We only prove the last claim:

$$\begin{aligned}
\Phi_{\pi(g_1)\underline{v}, \check{\pi}(g_2)\underline{v}'}(x) &= \left\langle \pi(x)\pi(g_1)\underline{v},^t \pi(g_2^{-1})\underline{v}' \right\rangle \\
&= \left\langle \pi(g_2^{-1})\pi(x)\pi(g_1)\underline{v}, \underline{v}' \right\rangle \\
&= \left\langle \pi(g_2^{-1} x g_1)\underline{v}, \underline{v}' \right\rangle \\
&= \Phi_{\underline{v}, \underline{v}'}(g_2^{-1} x g_1).
\end{aligned}$$

□

REMARK 17. We see that abstract representations have *concerete models*.

DEFINITION 18. Call an irrep (π, V) *discrete series* if it is isomorphic to an irreducible subrepresentation of the regular representaiton of G.

EXAMPLE 19. Suppose (π, V) is *unitarizable*, in that there is a G-invariant continuous Hermitian product on V (so that the completion is a Hilbert space). Equipping V' with the dual inner product, which is also invariant, we see that the matrix coefficients of π are bounded.

1.3. Compact groups: the Peter–Weyl Theorem

In this section G is a compact group, equipped with its probability Haar measure $\mathrm{d}g$.

1.3.1. Finite-dimensional representations: Schur orthogonality. Fix a representation (π, V) of G where V is finite-dimensional.

LEMMA 20 (Unitarity). *There is a G-invariant Hermitian product on V.*

PROOF. Let $(\cdot,\cdot)$ be any Hermitian product on V, and for $\underline{u}, \underline{v} \in V$ set

$$\langle \underline{u}, \underline{v} \rangle = \int_G (\pi(g)\underline{u}, \pi(g)\underline{v})\, \mathrm{d}g$$

where $\mathrm{d}g$ is the probability Haar measure on G. □

COROLLARY 21. *Let $W \subset V$ be an invariant subspace. Then it has a* complement*: another invariant subspace $W^\perp$ such that $V = W \oplus W^\perp$.*

PROOF. Take the orthogonal complement wrt an invariant Hermitian product. □

The following should be compared with the spectral theorem.

THEOREM 22 (Maschke). *Every finite-dimensional representation is a direct sum of irreducible subspaces.*

PROOF. Let $U \subset V$ be maximal wrt inclusion among all subspaces which are direct sums of irreducibles. If $U \neq V$ then $U^\perp$ is non-trivial; let $W \subset U^\perp$ be a non-zero invariant subspace of minimal dimension. Then W is necessarily irreducible and $U \oplus W$ is the direct sum of irreducibles, a contradiction. □

PROBLEM 23. Isomorphism as abstract, or as unitary, representations?

PROPOSITION 24 (Schur's Lemma). *Let $(\pi, V), (\sigma, W)$ be finite-dimensional irreducible representations of G. Then* $\mathrm{Hom}_G(V, W) \simeq \begin{cases} \mathbb{C} & \pi \simeq \sigma \\ 0 & \pi \not\simeq \sigma \end{cases}$.

PROOF. Since the kernel and image of an intertwining operator are invariant subspaces, any non-zero G-homomorphism from an irrep is injective and to an irrep is surjective. In particular, if π, σ are non-isomorphic they support no non-zero maps between them. It remains to compute $\mathrm{Hom}_G(V, V)$. For this let $T \in \mathrm{Hom}_G(V, V)$, so that $\pi(g)T = T\pi(g)$ for all $g \in G$. Since $\mathbb{C}$ is algebraically closed, T has at least one eigenvalue λ; let $V_\lambda = \mathrm{Ker}(T - \lambda)$, a non-trivial subspace of V. Then for any $\underline{v} \in V_\lambda$ we have $(T - \lambda)(\pi(g)\underline{v}) = \pi(g)((T - \lambda)\underline{v}) = \underline{0}$ so that $\pi(g)\underline{v} \in V_\lambda$ as well. It follows that $V_\lambda \subset V$ is a G-invariant subspace, and hence that $V_\lambda = V$ and $T = \lambda\,\mathrm{Id}$. □

Now let (π, V) be finite-dimensional. Every matrix cofficient of π is a continouos function on the compact space G, hence square-integrable.

PROPOSITION 25 (Schur Orthogonality). *Let $\pi, \sigma \in \mathrm{Rep}(G)$ be finite-dimensional irreps.*

(1) *Any two matrix coefficients of π, σ are orthogonal if π, σ are non-isomorphic.*

(2) *Let $d_\pi = \dim V_\pi$. Then for any $\underline{v}, \underline{w} \in V$ and $\underline{v}', \underline{w}' \in V'$ we have*

$$\left\langle \Phi^\pi_{\underline{u},\underline{u}'}, \Phi^\pi_{\underline{v},\underline{v}'} \right\rangle_{L^2(G)} = \frac{1}{d_\pi} \left\langle \underline{v}, \underline{u}' \right\rangle \left\langle \underline{u}, \underline{v}' \right\rangle$$

PROOF. Let $T \colon V \to W$ be any linear map, and let

$$\bar{T} = \int_G \sigma(g^{-1}) T \pi(g) \, \mathrm{d}g \,.$$

Then

$$\begin{aligned} \bar{T}\pi(h) &= \int_G \sigma(g^{-1}) T \pi(gh) \, \mathrm{d}g \\ &= \int_G \sigma(hg^{-1}) T \pi(g) \, \mathrm{d}g \\ &= \sigma(h) \bar{T} \,. \end{aligned}$$

It follows that $\bar{T} \in \mathrm{Hom}_G(V, W)$. Next, for any $\underline{v} \in V$, $\underline{v}' \in V'$, $\underline{w} \in W$, $\underline{w}' \in W'$ let $T = |\underline{w}\rangle \langle \underline{v}'|$. Then

$$\begin{aligned} \left\langle \underline{w}' \middle| \bar{T} \middle| \underline{v} \right\rangle &= \int \left\langle \underline{w}' \middle| \sigma(g^{-1}) \middle| \underline{w} \right\rangle \left\langle \underline{v}' \middle| \pi(g) \middle| \underline{v} \right\rangle \mathrm{d}g \\ &= \int_G \mathrm{d}g \overline{\left\langle \underline{w} \middle| \sigma(g) \middle| \underline{w}' \right\rangle} \left\langle \underline{v}' \middle| \pi(g) \middle| \underline{v} \right\rangle \\ &= \left\langle \Phi^\sigma_{\underline{w}',\underline{w}}, \Phi^\pi_{\underline{v},\underline{v}'} \right\rangle_{L^2(G)} , \end{aligned}$$

where we have identified W' with W via the Riesz representation theorem and the inner product.

(1) Suppose π, σ are non-isomorphic. Then $\bar{T} = 0$ and the two matrix coefficients are orthogonal.

(2) Suppose $V = W$, $\pi = \sigma$. Then $\bar{T} = \lambda \, \mathrm{Id}$ for some $\lambda \in \mathbb{C}$. Normalizing the Haar measure on G to be a probability measure, we see that $\bar{T}$ is the average of conjugates of T so

$$d_\pi \lambda = \mathrm{Tr}\, \bar{T} = \mathrm{Tr}\, T = \left\langle \underline{v}', \underline{w} \right\rangle .$$

Solving for λ it follows that

$$\begin{aligned} \left\langle \Phi^\pi_{\underline{w}',\underline{w}}, \Phi^\pi_{\underline{v},\underline{v}'} \right\rangle_{L^2(G)} &= \left\langle \underline{w}' \middle| \bar{T} \middle| \underline{v} \right\rangle = \lambda \left\langle \underline{w}' \middle| \mathrm{Id} \middle| \underline{v} \right\rangle \\ &= \frac{1}{d_\pi} \left\langle \underline{w}', \underline{v} \right\rangle \left\langle \underline{v}', \underline{u} \right\rangle . \end{aligned}$$

□

COROLLARY 26. $\langle \chi_\pi, \chi_\sigma \rangle_{L^2(G)} = \delta_{\pi \simeq \sigma}$.

COROLLARY 27. *For each finite-dimensional irrep π let $\mathcal{C}(\pi)$ be the space of matrix coefficients of π. Then*

$$\bigoplus_\pi \mathcal{C}(\pi) \subset L^2(G)$$

is an orthogonal direct sum.

1.3.2. Infinite-dimensional representations and the Peter–Weyl Theorem. Let (π, V) be a continuous representation of the locally compact group G on the quasi-complete locally convex TVS V.

LEMMA-DEFINITION 28. *TFAE for $\underline{v} \in V$, in which case we call it G-finite*

(1) $\mathrm{Span}_{\mathbb{C}} \{\pi(g)\underline{v}\}_{g \in G} \subset V$ *is finite-dimensional.*
(2) *There is a finite-dimensional G-invariant subspace $W \subset V$ with $\underline{v} \in W$.*

Furthermore, the set V_K of K-finite vectors is a G-invariant algebraic subspace of V.

PROOF. Given (1), set $W = \mathrm{Span}_{\mathbb{C}} \{\pi(g)\underline{v}\}_{g \in G}$ to get (2). Given (2), $\mathrm{Span}_{\mathbb{C}} \{\pi(g)\underline{v}\}_{g \in G} \subset W$ for all G-invariant subspaces W containing $\underline{v}$. Finally, if $\underline{v}_1, \underline{v}_2 \in V_K$, say with $\underline{v}_i \subset W_i$ with W_i G-inv't and f.d. then $\alpha \underline{v}_1 + \pi(g)\underline{v}_2 \in W_1 + W_2$ which is G-inv't and f.d. $\square$

PROPOSITION 29. *In a compact group G we have $\bigoplus_\pi \mathcal{C}(\pi) = L^2(G)_K$, where G acts on $L^2(G)$ via the right-regular representation $(R_g f)(x) = f(xg)$.*

PROOF. Since each $\mathcal{C}(\pi)$ is finite-dimensional, their algebraic direct sum is contained in $C(G)_K \subset L^2(G)_K$. Conversely, let $W \subset L^2(G)$ be a right-G-invariant finite-dimensional subpsace. By Maschke's Theorem 22, W is the direct sum of irreducible subpsaces so without loss of generality it suffices to show $W \subset \bigoplus_\pi \mathcal{C}(\pi)$ for irreducible W.

Now let $\{f_i\}_{i=1}^d \subset W$ be an orthonormal basis. Then for $f \in W$ and $g \in G$ we have $R_g f \in W$ and hence

$$R_g f = \sum_{i=1}^{d} a_i(g) f_i$$

for some $a_i(g) \in \mathbb{C}$. In fact,

$$a_i(g) = \left\langle f_i, R_g f \right\rangle_{L^2(G)} = \Phi^W_{f_i,f}(g) \in \mathcal{C}(W)$$

and we conclude that for *fixed* g

$$R_g f = \sum_{i=1}^{d} \Phi^W_{f_i,f}(g) f_i$$

(the sum in $W \subset L^2(G)$). In other words, given g it holds for almost every $x \in G$ that

$$f(xg) = \sum_{i=1}^{d} \Phi^W_{f_i,f}(g) f_i(x) .$$

If the identity held for *all* x we could set $x = e$ and write f as a linear combination of matrix coefficients. To get around this difficulty consider both sides as functions on $G \times G$. Now both sides are in $L^2(G \times G)$, so by Fubini they are equal a.e. Applying Fubini in the other order it follows that for almost every $x \in G$ we have $f(xg) = \sum_{i=1}^{d} \Phi^W_{f_i,f}(g) f_i(x)$ for almost every $g \in G$, and that is the desired claim. $\square$

DEFINITION 30. For $f \in C_c(G)$ and $\underline{v} \in V$ set $\pi(f)\underline{v}$ by

$$\pi(f)\underline{v} = \int_G f(g)\pi(g)\underline{v}\, \mathrm{d}g .$$

LEMMA 31. *$\pi(f)\colon V \to V$ is a continuous linear map, and $f \mapsto \pi(f)$ is a continuous algebra homomorphism $C_c(G) \to \mathrm{End}(V)$ where $C_c(G)$ is equipped with the convolution product and the direct limit topology.*

PROOF. Scaling, we may assume $|f(g)| \leq 1$ for all g. Let $U \subset V$ be a closed convex neighbourhood of zero. Then for each $g \in \mathrm{supp}(f)$ there are neighbourhoods $g \in W_g \subset G$ and (convex) $\underline{0} \in U_g \subset V$ such that $\pi(x)\underline{u} \in \frac{1}{\mathrm{vol}\,\mathrm{supp}(f)} U$ for all $x \in W_g$, $\underline{u} \in U_g$. Covering $\mathrm{supp}(f)$ with $\cup_{i=1}^r W_{g_i}$ and setting $\bar{U} = \cap_{i=1}^r U_{g_i}$ we see that for all $g \in \mathrm{supp}(g)$ and $\underline{v} \in \bar{U}$, $f(g)\pi(g)\underline{v} \in \frac{1}{\mathrm{vol}\,\mathrm{supp}(f)} U$. It follows that $\pi(f)\underline{v} \in U$.

Rest proved similarly. □

COROLLARY 32. *Let $\{f_n\} \subset C_c(G)$ be an approximate identity. Then $\pi(f_n)\underline{v} \to \underline{v}$.*

EXAMPLE 33 (Smoothing). Let $V \subset L^2(G)$ be a closed G-invariant subspace. Then $V \cap C(G)$ is dense in G.

PROOF. It suffices to show that $\pi(f)\varphi \in C(G)$ for each $f \in C_c(G)$, $\varphi \in L^2(G)$. Indeed,

$$\begin{aligned} (\pi(f)\varphi)(x) &= \int f(g)\varphi(g^{-1}x)\,\mathrm{d}g \\ &= \int f(xg)\varphi(g^{-1})\,\mathrm{d}g \end{aligned}$$

so that

$$\begin{aligned} |(\pi(f)\varphi)(x) - (\pi(f)\varphi)(y)| &= \left| \int \delta(g)\left(f(xg^{-1}) - f(yg^{-1})\right)\varphi(g)\,\mathrm{d}g \right| \\ &\leq \left\| \delta(g)\left(f(xg^{-1}) - f(yg^{-1})\right) \right\|_{L^2(G)} \|\varphi\|_{L^2(G)} \\ &\xrightarrow[y \to x]{} 0 \end{aligned}$$

since f is uniformly continuous and δ is bounded on any compact set.

Suppose now that G is compact. □

THEOREM 34 (Peter–Weyl I). *We have*

$$L^2(G) = \hat{\bigoplus_{\pi}} \mathcal{C}(\pi).$$

PROOF. Let $V = \left(\bigoplus_\pi \mathcal{C}(\pi)\right)^\perp$ and note that V is a subrepresentation of $\left(L^2(G), R\right)$. If $V \neq \{0\}$ let $f \in V$ be non-zero, and by continuity of the G-action on $L^2(G)$ let $U \subset G$ be a symmetric, conjugation-invariant neighbourhood of 1 such that $\|R_u f - f\|_2 \leq \frac{1}{2}\|f\|$. Let $\chi \in C_c(U)$ be positive, satisfy $\chi(u) = \chi(u^{-1})$, integrate to 1 and be conjugation invariant. Then $\|R(\chi)f - f\|_2 \leq \frac{1}{2}\|f\|$ and in particular $R(\chi)\colon V \to V$ is a non-zero operator. It is also self-adjoint and compact. By the spectral theorem its eigenspaces are finite-dimensional and it follows that V contains G-finite vectors, a contradiction. □

COROLLARY 35 (Peter–Weyl II). *$\bigoplus_\pi C_c(\pi)$ is dense in $C(G)$.*

PROOF. Since the matrix coeffs of the tensor product are products, this is a subalgebra closed under complex conjugation and it suffices to show it separates the points. By G-invariance it suffices to separates ponts from the identity.

For this consider $\bigcap_\pi \mathrm{Ker}(\pi)$. Every $f \in L^2(G)$ is invariant by this closed subgroup, so it's trivial. It follows that for any $g \in G$ there is π such that $\pi(g) \neq \mathrm{id}$. Let $\underline{v} \in V_\pi$ be of norm 1 such that $\pi(g)\underline{v} \neq \underline{v}$. Then by unitarity $\langle \pi(g)\underline{v}, \underline{v} \rangle \neq 1$ and hence

$$\Phi_{\underline{v},\underline{v}}(g) \neq 1\,.$$

□

THEOREM 36 (Peter–Weyl II). *Every irrep of G is finite-dimensional; for any representation V_K is dense in V.*

PROOF. Clearly the second assertion implies the first. We first note that the argument of Theorem 34 shows that $\{\pi(\chi)\underline{v} \mid \underline{v} \in V, \chi \in C(G)\}$ is dense in V. Now to see that V_K is dense in V it suffices to show that $V_K = \{\pi(f)\underline{v} \mid \underline{v} \in V, f \in C(G)_G\}$ is dense. For that note that for any neighbourhood $W \subset V$ of zero, if f is ε-close to χ for ε small enough then $\pi(f-\chi)\underline{v} \in W$. □

CHAPTER 2

Lie Groups and Lie Algebras

2.1. Smooth manifolds

2.1.1. Manifolds.

DEFINITION 37. Let $U \subset \mathbb{R}^n$ be open. Then $C^\infty(U;\mathbb{R}^m)$ is the set of infinitely differentiable $\mathbb{R}^m$-valued functions on U.

DEFINITION 38. A *coordinate chart* (or *patch)* in a topological space M is a pair (U,φ) where $U \subset M$ is open and $\varphi\colon U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$. Two coordinate patches $(U_1,\varphi_1),(U_2,\varphi_2)$ are *compatible* if $\varphi_1 \restriction_{U_1\cap U_2} \circ (\varphi_2 \restriction_{U_1\cap U_2})^{-1}$ is a smooth map.

An *altas* on M is a covering of M by compatible coordinate patches. A *smooth manifold* is a pair $(M,\mathcal{A})$ where M is a second countable topological space and $\mathcal{A}$ is an atlas on M.

EXAMPLE 39. $\mathbb{R}^n$, S^n, $\mathbb{T}^n$.

LEMMA 40. *If two charts are compatible with an atlas they are compatible with each other.*

COROLLARY 41. *Every atlas is contained in a maximal atlas, namely the set of all charts compatible with the given atlas.*

DEFINITION 42. A maximal atlas is also known as a *smooth structure* on M.

EXAMPLE 43. Exotic spheres.

LEMMA 44. *If $m \neq n$ $\mathbb{R}^m,\mathbb{R}^n$ are not locally homeomorphic so for a connected manifold the dimension need not be assumed constant.*

DEFINITION 45. Let M,N be smooth manifolds. A map $f\colon M^m \to N^n$ is *smooth* if for every charts (U,φ) of M and (V,ψ) of N, $\psi\circ f\circ\varphi^{-1}$is smooth.

LEMMA 46. *Composition of smooth maps is smooth.*

2.1.2. Tangent and contagent spaces.

Fix a vector space k.

DEFINITION 47. A *Lie algebra* over k is a k-vector space $\mathfrak{g}$ together with a bilinear form $[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying:

(1) (alternating)$[X,X]=0$
(2) (Jacobi identity) $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$.

EXAMPLE 48 (Standard constructions). Let A be an associative k-algebra. We get two natural Lie algebras from it:

(1) A itself, equipped with $[a,b]=ab-ba$.
(2) Call $d\in\mathrm{End}_{k\text{-vsp}}(A)$ a *derivation* if $d(ab)=d(a)b+ad(b)$. Then the space $\mathcal{D}_A$ of derivations is a Lie subalgebra of $\mathrm{End}_{k\text{-vsp}}(A)$.

(3) One canonical example: $A = C^\infty(M)$; then $\mathcal{D}_M \stackrel{\text{def}}{=} \mathcal{D}_{C^\infty(M)}$ is called the set of (smooth) *vector fields* on M.

LEMMA 49 (Localization of vector fields). *Let $X \in \mathcal{D}_M$, $f,g \in C^\infty(M)$.*

(1) *Let f be constant. Then $Xf \equiv 0$.*
(2) *Let $f(p) = 0$. Then $\left(Xf^2\right)(p) = 0$.*
(3) *Let f be constant in a neighbourhood of p. Then $(Xf)(p) = 0$. In particular, if $f = g$ in a neighbourhood of p then $Xf(p) = Xg(p)$.*

PROOF. Say $f(p) = 1$ for all x. Then $Xf = X\left(f^2\right) = 2f \cdot Xf = 2Xf$. It follows that $Xf \equiv 0$. Simliarly, if $f(p) = 0$ then $\left(Xf^2\right)(p) = 2f(p)Xf(p) = 0$.

Let U be a neighbourhood of $p \in U$ and suppose $f \restriction_U \equiv 1$. Choose $g \in C_c^\infty(U)$ such that $g(p) \neq 0$. Since $fg = g$ we have $Xf \cdot g + f \cdot Xg = Xg$, Evaluating at p we get $Xf(p)g(p) = 0$ so $Xf(p) = 0$. □

LEMMA-DEFINITION 50. *$I_p = \{f \in C^\infty(M) \mid f(p) = 0\}$ is a maximal ideal of $C^\infty(M)$.*

LEMMA-DEFINITION 51 (Hadamard). *The* contagent space $T_p^*M = I_p/I_p^2$ *is a vector space of dimension n and $\bigcup_{p\in M} T_p^*M$ is a vector bundle, the* contagent bundle.

PROOF. Let f vanish in a neighbourhood U of p, and let $g \in C_c^\infty(U)$ vanish at p as well. Then $f = fg \in I_p^2$. It follows that $f,g \in C^\infty(M)$ agree in a neighbourhood of p then $f - g \in I_p^2$. We can now work locally, in particular near $\underline{0} \in \mathbb{R}^n$. We next show that every class in I_p/I_p^2 has a linear representative. Indeed let f be smooth in a neighbourhood of $\underline{0} \in \mathbb{R}^n$ and set $g(t) = f(t\underline{x})$. Then

$$\begin{aligned} f(\underline{x}) - f(\underline{0}) = g(t) - g(1) &= \int_0^1 g(t)\,\mathrm{d}t \\ &= \int_0^1 \underline{x} \cdot \nabla f(t\underline{x})\,\mathrm{d}t \\ &= \sum_{i=1}^n x_i \cdot \int_0^1 \frac{\partial}{\partial x^i} f(t\underline{x})\,\mathrm{d}t \\ &= \nabla f(\underline{0}) \cdot \underline{x} + \sum_{i=1}^n x_i h_i \end{aligned}$$

where $h_i(\underline{x}) = \int_0^1 \frac{\partial}{\partial x^i} f(t\underline{x})\,\mathrm{d}t - \frac{\partial f}{\partial x^i}(\underline{0}) \in I_{\underline{0}}$. It follows that

$$f(\underline{x}) - f(\underline{0}) - \nabla f(\underline{0}) \cdot \underline{x} \in I_p^2.$$

To see that the linear functions inject into I_p/I_p^2 (so that the dimension is n) note that each linear function has a non-zero directional derivative, but that operation is a derivation in $C^\infty(U)$ ($U \subset \mathbb{R}^n$) and vanishes on elements of I_p^2. □

LEMMA-DEFINITION 52 (The tangent space). *The linear dual $T_pM = \mathrm{Hom}_{\mathbb{R}}\left(T_p^*M, \mathbb{R}\right)$ is called the* tangent space. The resulting bundle is called the *tangent bundle*.

(1) *The pairing $(X,f) \mapsto Xf(p)$ associates to each vector field X a linear functional on T_p^*X.*
(2) *The resulting map $\mathcal{D}_M \to \left(T_p^*M\right)'$ is surjective.*

CONCLUSION 53. In local coordinates, a vector field is an operator of the form $\sum_{i=1}^n a_i(\underline{x}) \frac{\partial}{\partial x^i}$.

EXERCISE 54. T_pM is also the space of derivations on the algebra of germs of smooth functions at p.

PROPOSITION 55 (Canonical sheaf).

(1) *Let X be a vector field on M, $U \subset M$ an open set. For $f \in C^\infty(U)$ and $p \in U$ let $h \in C^\infty_c(U)$ such that $h \equiv 1$ near p and set $(X \restriction_U f)(p) = (X(hf))(p)$ (note that $hf \in C^\infty_c(M)$). Then $X \restriction_U$ is a well-defined vector field on U and $X \mapsto X \restriction_U$ is a map of lie algebras.*
(2) *(Patching) Let $\{U_i\}_{i \in I}$ be an open cover of M. Let X, Y be a vector fields on M and suppose that $X \restriction_{U_i} = Y \restriction_{U_i}$ for all i then $X = Y$.*
(3) *(Gluing) Let $\{U_i\}_{i \in I}$ be an open cover of M and suppose given for each i a vector field X_i on U_i such that $X_i \restriction_{U_i \cap U_j} = X_j \restriction_{U_i \cap U_j}$ for all i, j. Then there is a vector field X on M such that $X_i = X \restriction_{U_i}$.*

2.1.3. Derivatives of maps.

LEMMA-DEFINITION 56. *Let $\varphi \colon M \to N$ be a smooth map. Let $p \in M$ and $v \in T_pM$. Then the map $d\varphi_p(v) \colon C^\infty(N) \to \mathbb{R}$ given by $f \mapsto v(f \circ \varphi)$ is a local derivation at $\varphi(p)$. It is called the* differential *of φ. The map $d\varphi_p \colon T_pM \to T_{\varphi(p)}N$ is linear and extends to a smooth map $d\varphi \colon TM \to TN$ compatible with φ. The construction is functorial (in other words, the chain rule holds).*

THEOREM 57 (Inverse and implicit function theorems). *Let $\varphi \colon M \to N$ be smooth.*

(1) *Suppose $d\varphi_p$ is injective. Then φ is injective in a neighbourhood of p.*
(2) *Suppose $d\varphi_p$ is a surjective. Then φ is an open map in a neighbourhood of p.*
(3) *Suppose $d\varphi_p$ is an isomorphism. There are open neighbourhoods of p and $\varphi(p)$ for which φ is a diffeomorphism.*
(4) *Suppose $d\varphi_p$ is surjective for p on a level set $P = \varphi^{-1}(n)$. Then the level set is a submanifold of dimension* $\dim N - \dim M$.

DEFINITION 58. A smooth map $f \colon M \to N$ is a:

(1) *Submersion* if df_p is injective for every $p \in M$.
(2) *Local embedding* if it is a submersion and for every $p \in U \subset M$ there is $f(p) \in V \subset N$ such that $f \restriction_{f^{-1}(V)}$ is a homeomorphism onto its image (with the relative topology)
(3) An embedding if it is an injective immersion which is a homeomorphism onto its image.
(4) A *diffeomorphism* if it has a smooth inverse.

DEFINITION 59. A *parametrized submanifold* of N is a pair (M, f) where $f \colon M \to N$ is an injective submersion. Two parametrizations $(M_1, f_1), (M_2, f_2)$ are *equivalent* if they are conjugate by a diffeomorphism of M_1, M_2. A *submanifold* of N is an equivalence class.

If (M, f) is a parametrized submanifold N then $T(N) = f_*(TM)$ is a subbundle of $TN \restriction_M$. It is independent of the choice of parametrization. Conversely, we'd like to investigate when a choice of subspace of T_pM at each p corresponds to a submanifold.

DEFINITION 60. Let $\gamma \colon [a,b] \to M$ be a smooth curve. We then set $\dot\gamma(t) = d\gamma\left(\frac{\partial}{\partial t}\right)(t)$. We say γ is an *integral curve* of $X \in \mathcal{D}_M$ if $\dot\gamma(t) = X(\gamma(t))$ for each t.

- The Picard Theorem on ODE shows that for any X and $p \in M$ there is an integral curve of X through p living on an interval about 0, and that any two integral curves with $\gamma(0) = p$ agree on their interval of definition.

We now generalize this from 1-dimensional submanifolds to higher dimension.

DEFINITION 61. A *distribution* of dimension k on M is equivalently either of:

1. A smooth choice of k-dimensional subspaces $V_p \subset T_pM$ for each $p \in M$.
2. A smooth section of the Grassmanian bundle, or a subbundle of TM.
3. For a covering set of neighbourhoods $U \subset M$ choices of vector fields $\{X_i\}_{i=1}^k \subset \mathcal{D}_U$ so that for each $p \in U$, $\{X_i(p)\} \subset T_pM$ are linearly independent and so that $V_p = \mathrm{Span}_{\mathbb{R}}\{X_i(p)\}_i$ is independent of U as long as $p \in U$.

Call a vector field $X \in \mathcal{D}_M$ a *section* of the distribution V if (1) $X_p \in V_p$ for each p iff (2) it is a section of the subbundle (3) For each U there are $a_i \in C^\infty(U)$ so that $X \restriction_U = \sum_i a_i X_i$.

DEFINITION 62. Call a submanifold $\left(N^k, \varphi\right)$ of M *tangent* to the distribution V if for each $p \in N$, $d\varphi_p$ is an isomorphism of T_pN and $V_{\varphi(p)} \subset T_{\varphi(p)}M$.

OBSERVATION 63. *Suppose $N \subset M$ is tangent to V, and let X,Y be sections of V. We can then think of X,Y as vector fields on N, so that $[X,Y]$ is a vector field on N as well. It follows that $[X,Y]$ is also a section of V.*

In fact, this necessary condition is also sufficient:

THEOREM 64 (Frobenius). *The following are equivalent for a distribution V on M:*

1. *Through each $p \in M$ there is a unique (up to equivalence) submanifold tangnet to V; this submanifold is injectively submersed.*
2. *The distribution is* completely integrable*: for every two sectoins X,Y of V, the vector field $[X,Y]$ is also a section.*

REMARK 65. In the local view above it suffices to check that the integrability condition on the spanning fields: $\left[X_i, X_j\right] = \sum_k a_k X_k$ for some $a_k \in C^\infty(U)$.

2.2. Lie groups

DEFINITION 66. A *Lie group* is a group object in the category of smooth manifolds, in other words a smooth manifold G together with smooth maps $\cdot\colon G \times G \to G$ and ${}^{-1}\colon G \to G$ such that $\left(G, \cdot, {}^{-1}\right)$ is an abstract group. A *homomorphism* of Lie group is an abstract homomorphism which is also a smooth map.

EXAMPLE 67. The basic example is $\mathbb{R}$, but we also have:

1. $\mathbb{R}^n$, $(\mathbb{R}/\mathbb{Z})^n = \mathbb{R}^n/\mathbb{Z}^n$
2. $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{R})$
3. $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{SO}(Q) = \mathrm{SO}(p,q)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$
4. Direct and semidirect products.
5. $\mathrm{Isom}(\mathbb{E}^n)$, $\mathrm{Isom}(M,g)$
6. $\mathrm{Aff}_n(\mathbb{R})$

DEFINITION 68. An *action* of a Lie group G on a smooth manifold M is a smooth map $\cdot\colon G \times M \to M$ which is a group action.

DEFINITION 69. A *Lie subgroup* H of the Lie group G is a subgroup $H < G$ which is also a submanifold, in other words the image of an injective immersion of Lie groups.

EXAMPLE 70. Lie of irrational slope on a torus.

REMARK 71. There is some play in the joints here.

(1) Enough to assume C^2, and may assume real-analytic (any C^2 structure is compatible with a unique smooth, even real-analytic, structure).
(2) Sophus Lie actually considered local Lie group actions.

2.3. Lie Algebras and the exponential map

2.3.1. Lie algebra. The Lie group G acts on itself by left multiplication. This *regular action* is a smooth action. In particular each $g \in G$ acts by translation on the set of vector fields of G, and we call a vector field X *left-invariant* if $g \cdot X = X$. Recall that for any manifold we have a surjective map $\{\mathcal{D}_M\} \to T_pM$.

LEMMA 72. *Restricting this map to the left-invariant vector fields on G gives a linear isomorphism* $\{$*left-invariant vector fields on G*$\} \to T_eG$*.*

PROOF. For the inverse map, for any manifold M a smooth action of G on M extends to a smooth action on TM by $g \cdot (p,v) = (gp, dg(v))$ where dg is the derivative of the map $g\cdot\colon M \to M$. In particular, G acts on TG . Now for $v \in T_eG$ the orbit $g \mapsto g \cdot (e,v)$ is a smooth left-invariant vector field. □

Note that if X,Y are left-invariant so is $[X,Y]$.

DEFINITION 73. The *Lie algebra* of G is the Lie algebra of left-invariant vector fields, equivalently the same Lie algebra realized as the tangent space T_eG. We write $\mathfrak{g} = \mathrm{Lie}(G)$ for the Lie algebra.

THEOREM 74. *If* $f \in \mathrm{Hom}(G,H)$ *then* $df\colon \mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra homomorphism.*

LEMMA 75. *A connected Lie group is generated by any open subset*

THEOREM 76. *Every subalgebra exponentiates to a subgroup*

PROOF. The distribution defined by the subalgebra is integrable, so apply Frobenius. The leaf through the origin is self-invariant, hence a subgroup. □

2.3.2. Exponential map.

LEMMA-DEFINITION 77. *The integral curves through left-invariant vector fields live forever. Write the integral curve through* $X \in \mathfrak{g}$ *as* $t \mapsto \exp(tX)$*.*

REMARK 78. Uniqueness of integral curves shows that indeed this only depends on $tX \in \mathfrak{g}$ rather than on t,X separately.

PROPOSITION 79. $\exp\colon \mathfrak{g} \to G$ *is a local diffeomorphism with derivative* Id.

PROOF. Differentiate ODE and inverse function theorem □

LEMMA 80. *Homomorphisms repsect the exponential map*

PROOF. $f(\exp(tX))$ is an integral curve of $df(X)$. □

PROPOSITION 81. *Exponential map of* $\mathrm{GL}_n(\mathbb{R})$ *is given by the matrix exponential.*

2.4. Closed Subgroups

THEOREM 82 (Cartan 1930). *Let $H < G$ be a closed subgroup. Then H is a Lie subgroup (in particular, a submanifold of G).*

PROOF. Let $\mathfrak{h} = \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R} : \exp(tX) \in H\}$. Then $\mathfrak{h} \subset \mathfrak{g}$ is closed under scaling. Also, $Z(t) = \log(\exp(tX)\exp(tY))$ has $Z(0) = 0$, $Z'(0) = X + Y$ so $Z(t) = t(X + y) + O(t^2)$ and it follows that $\exp(t(X+Y)) \in H$. Now $\exp \colon \mathfrak{h} \to H$ is locally bijective (use local exponential coordinateS) □

LEMMA 83. *Let $V \subset G$ be a small enough neighbourhood of the identity. Then*

THEOREM 84. *Let H be a closed connected subgroup of G. Then G/H has a unique manifold structure such that $\pi \colon G \to G/H$ is smooth. Furthermore, the regular action of G on G/H is a Lie group action.*

THEOREM 85. *A map of Lie groups is a covering iff its derivative is an isomorphism*

PROOF. A covering map is a local diffeo, hence gives isom of Lie algebras. Conversely, let $df = df_e$ be an isomorphism for $f \colon G \to H$. By homogeneity df_g is injective for each $g \in G$ so f is a local diffeomorphism. The kernel $\Gamma = \mathrm{Ker}(f)$ is a closed subgroup, which is zero-dimensional hence discrete. Let $U \subset G$ be a small enough neighbourhood so that its translates by Γ are disjoint and such that $f \restriction_U$ is a diffeo. Then $f^{-1}(f(U)) \simeq \Gamma \times U$. □

THEOREM 86. *Let $df \colon \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected and H is connected then this lifts to f.*

PROOF. Realize the graph of f as a subgroup of $G \times H$ corresponding to a lie subalgebra. Projection on G, so the graph is a function. □

THEOREM 87 (Ado). *Every finite-dimensional Lie algebra has a faithful representation into $\mathfrak{gl}_n(\mathbb{R})$.*

PROOF. Adjoint gives a representation mod centre □

COROLLARY 88. *Every Lie algebra is the Lie algebra of some group.*

THEOREM 89. *Let $H < G$ be a closed subgroup. Then G/H has a unique manifold structure such that the quotient map is smooth.*

PROOF. Local exponential coordinates. □

2.5. The adjoint representation

DEFINITION 90. Let $g \in G$. Then $\mathrm{Ad}_g \colon G \to G$ given by $\mathrm{Ad}_g(x) = gxg^{-1}$ is an automorphism, in particular a group homomorphism. We also write Ad_g for its derivative, $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$.

LEMMA 91. $\mathrm{Ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ *is a smooth representation.*

DEFINITION 92. Write $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ for the derivative of the adjoint representation.

THEOREM 93. $\mathrm{ad}_X \cdot Y = [X,Y]$.

PROOF. Since ad is the derivative of Ad, $\exp(t\,\mathrm{ad}_X) = \mathrm{Ad}_{\exp(tX)}$ in $\mathrm{GL}(\mathfrak{g})$. It follows that

$$\begin{aligned}
\mathrm{ad}_X \cdot Y &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(tX)} Y \\
&= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \mathrm{Ad}_{\exp(tX)} \exp(sY) \\
&= \frac{d}{ds}\Big|_{s=0} \frac{d}{dt}\Big|_{t=0} \exp(tX)\exp(sY)\exp(-tX) \\
&= \frac{d}{ds}\Big|_{s=0} \left(X_e \cdot (\exp(sY))_* - X_{\exp(sY)}\right) .
\end{aligned}$$

More precisely, this means that for $f \in C^\infty(G)$,

$$\langle df_e, \mathrm{ad}_X \cdot Y \rangle = \frac{d}{ds}\Big|_{s=0} \left\langle df_{\exp(sY)}, X_e \cdot (\exp(sY))_* - X_{\exp(sY)} \right\rangle .$$

Now

$$\frac{d}{ds}\Big|_{s=0} \left\langle df_{\exp(sY)}, X_e \cdot (\exp(sY))_* \right\rangle = \frac{d}{ds}\Big|_{s=0} \left\langle d(R_{\exp(sY)} f)_e, X_e \right\rangle = \left\langle \frac{d}{ds}\Big|_{s=0} d\left(g \mapsto f(g\exp(sY))\right)_e, X_e \right\rangle = (XYf)($$

and

$$\frac{d}{ds}\Big|_{s=0} \left\langle df_{\exp(sY)}, X_{\exp(sY)} \right\rangle = (YXf)(e)$$

so we are done. □

COROLLARY 94. $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *is a Lie algebra representation:* $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$.

PROOF. This follows immediately from the Jacobi identity. □

COROLLARY 95. *Let $H < G$ be connected Lie groups. Then H is normal iff $\mathfrak{h}$ is a Lie ideal.*

PROOF. If H is normal then H is Ad-stable hence $\mathfrak{h}$ is Ad-stable hence $\mathfrak{h}$ is ad-stable. Conversely, for X close enough to the origin we have $\exp(\mathrm{ad}_X) = \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{ad}_X)^k$. Now if $\mathfrak{h}$ is ad_X-stable it follows that it is also $\exp(\mathrm{ad}_X)$-stable and hence $\mathrm{Ad}_{\exp X}$-stable. But by the group-algebra correspondence this means H is $\mathrm{Ad}_{\exp X}$-stable. Since the small X generate G we are done. □

COROLLARY 96. *Let G be connected. Then* $Z(G) = \ker(\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g}))$.

PROOF. $g \in G$ is central iff for all small enough X, $g \exp X g^{-1} = \exp X$ iff $\exp(\mathrm{Ad}_g X) = \exp X$ iff $\mathrm{Ad}_g X = X$. □

COROLLARY 97. *Let G be connected. Then $\mathfrak{g}$ is abelian iff G is abelian iff* $\exp\colon \mathfrak{g} \to G$ *is a surjective group homomorphism.*

PROOF. If $\mathrm{ad}_X = 0$ for all X then $\exp(\mathrm{ad}_X) = \mathrm{Id}$ for all X so a neighbourhood of the identity is contained in Ker(Ad). If G is abelian let $X, Y \in \mathfrak{g}$. Then $t \mapsto \exp(tX)\exp(tY)$ is a group homomorphism $\mathbb{R} \to G$. Since its derivative at $t = 0$ is $X + Y$ we conclude that $\exp(tX)\exp(tY) = \exp(t(X+Y))$. Now setting $t = 1$ shows that exp is a homomorphism, and since the image contains a generating set it's surjective. Finally, if exp is a surjective homomorphism then its image G is abelian. □

THEOREM 98. *A connected abelian Lie group is of the form $\mathbb{R}^n \times \mathbb{T}^n$.*

PROOF. Ker(exp) is a discrete subgroup of $\mathbb{R}^d$. □

CHAPTER 3

Compact Lie groups

3.1. Linearity

As an application of our representation theory of compact groups we get:

THEOREM 99. *Every compact Lie group has a faithful finite-dimensional representations. Equivalently, every compact group is isomorphic to a closed subgroup of some $U(n)$.*

PROOF. The representation of G on $L^2(G)$ is faithful. By Peter–Weyl it follows that $\bigcap_{\pi\in\hat{G}} \mathrm{Ker}(\pi) = \{e\}$. Let □

3.2. Characters and cocharacters of tori

Let $\mathbb{R}^n/\mathbb{Z}^n$, $\mathbb{R}^m/\mathbb{Z}^m$ be tori. We'd like to study $\mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$. The cases $m = 1$ (characters) and $m = n$ (automorphisms) are particularly important.

First, let $f\colon \mathbb{Z}^n \to \mathbb{Z}^m$ be a group homomorphism. Extending scalars gives a homomorphism $f_{\mathbb{R}} = f \otimes_{\mathbb{Z}} \mathbb{1}\colon \mathbb{Z}^n \otimes \mathbb{R} \to \mathbb{Z}^m \otimes \mathbb{R}$. Since $f_{\mathbb{R}}(\mathbb{Z}^n) = f(\mathbb{Z}^n) \subset \mathbb{Z}^m$, $f_{\mathbb{R}}$ descends to a homomorphism $\bar{f}\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^m/\mathbb{Z}^m$.

LEMMA 100. *The map $f \mapsto \bar{f}$ is an isomorphism* $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^m) \to \mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$.

PROOF. We need to construct the inverse map. For this let $\exp_n\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the quotient map, which is also the exponential map of this commutative Lie group with kernel $\mathbb{Z}^n$. Then given $\bar{f} \in \mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$ consider the linear map $d\bar{f} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. The usual identity $\bar{f}(\exp_n X) = \exp_m\left(d\bar{f}(X)\right)$ here reads $\exp_m \circ df = \bar{f} \circ \exp_n$, in other words that $d\bar{f}(\mathbb{Z}^n) \subset \mathbb{Z}^m$ and hence that $f = d\bar{f}\restriction_{\mathbb{Z}^n}$ is the desired element of $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^m)$. □

COROLLARY 101. $\mathrm{Aut}(\mathbb{T}^n) \simeq M_n(\mathbb{Z})^\times = \mathrm{GL}_n(\mathbb{Z})$. *In particular,* $\mathrm{Aut}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$ *is discrete.*

COROLLARY 102. $\widehat{\mathbb{T}^n} = \mathrm{Hom}\left(\mathbb{T}^n, S^1\right) = \{e(\underline{k}\cdot\underline{x})\}_{\underline{k}\in\hat{\mathbb{Z}}^n}$ *where* $\hat{\mathbb{Z}}^n = \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ *is the dual lattice, and* $e(z) = e^{2\pi i z}$.

PROOF. $z \mapsto e(z)$ is an isomorphism $\mathbb{R}/\mathbb{Z} \to S^1$. □

LEMMA 103. *Tori are topologically generated by single elements.*

PROOF. Let $\{1\} \cup \{\xi_i\}_{i=1}^n \subset \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then $\underline{\xi}$ is such an element. In fact (Weyl equidistribution) every orbit $\left\{\underline{x} + j\underline{\xi}\right\}_{j=1}^\infty$ is equidistributed in the torus. □

3.3. The exponential map

From now on let G be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra, and let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation. Since G is compact we may fix a G-invariant inner product (and associated Euclidean norm) on $\mathfrak{g}$.

LEMMA 104. *A connected compact Lie group has a bi-invariant Riemannian metric*

REMARK 105. The map $g \mapsto g^{-1}$ is an isometry of this metric. In other words, we have a *symmetric space*. (c.f. PS ??)

PROPOSITION 106. *Fix a bi-invariant metric on G. Then the Riemannian and Lie exponential maps agree.*

PROOF. Let $\gamma(t)$ be a Riemannian geodesic based at the origin. Then $t \mapsto \gamma(t_0 + t)$, $t \mapsto \gamma(t_0)\gamma(t)$ and $t \mapsto \gamma(t)\gamma(t_0)$ are also geodesics (because the group acts by isometries) which meet at $t = 0$ and have the same derivative at that time. It follows that $\gamma(t_0 + t) = \gamma(t_0)\gamma(t)$, that is that the geodesic is a one-parameter subgroup. $\square$

COROLLARY 107. *The exponential map of a connected compact Lie group is surjective.*

COROLLARY 108. *The intersection of two connected subgroups is connected.*

PROOF. The Lie algebra of the intersection is the intersection of the Lie algebras. $\square$

3.4. Maximal Tori

Fix a compact connected Lie group G. A *torus* in G is a subgroup T of G isomorphic to $\mathbb{T}^n$ for some n. Being compact tori are always closed.

LEMMA 109. *Every $g \in G$ is contained in a torus.*

PROOF. Suppose $g = \exp(X)$ for $X \in \mathfrak{g}$. Then $\{\exp(tX)\}_{t\in\mathbb{R}}$ is a connected commutative subgroup of G. Its closure is a connected commutative compact group, that is a torus. $\square$

LEMMA 110. *Let T be a torus in G, and let $\mathfrak{t}$ be its Lie algebra. Then:*

(1) $Z_G(T)$ *is connected.*
(2) $Z_G(\mathfrak{t}) = Z_G(T)$
(3) $\mathrm{Lie}\, Z_G(T) = Z_{\mathfrak{g}}(\mathfrak{t})$.
(4) $N_G(T)^\circ = Z_G(T)$.

PROOF.

(1) Let $t \in T$ generate a dense subgroup, so that $Z_G(t) = Z_G(T)$. Embed G in $U(n)$. Wlog the image of t is diagonal and then $Z_{U(n)}(t)$ is block-diagonal, in particular connected. It follows that $Z_G(T) = Z_G(t) = G \cap Z_{U(n)}(t)$ is connected.
(2) If $g \in Z_G(T)$ then $\mathrm{Ad}_g \in \mathrm{Aut}(T)$ being trivial means that $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{t})$ is trivial. Conversely, the exponential map of T is surjective and for any $H \in \mathfrak{t}$ and $g \in Z_G(\mathfrak{t})$ we have
$$\mathrm{Ad}_g(\exp H) = \exp(\mathrm{Ad}_g H) = \exp H\,.$$
(3) If $X \in Z_{\mathfrak{g}}(\mathfrak{t})$ then for any $s \in \mathbb{R}$, $\mathrm{Ad}_{\exp(sX)} \restriction_{\mathfrak{t}} = \exp(\mathrm{ad}_{sX} \restriction_{\mathfrak{t}}) = \exp(0) = \mathrm{Id}$ and hence $\exp(sX) \in Z_G(\mathfrak{t})$ and $X \in \mathrm{Lie}(Z_G(\mathfrak{t}))$. Conversely, suppose that $\mathrm{Ad}_{\exp(sX)} \in Z_G(\mathfrak{t})$ for all s. Differentiating with respect to s we get that $\mathrm{ad}_X \restriction_{\mathfrak{t}} = 0$ that is that $X \in Z_{\mathfrak{g}}(\mathfrak{t})$.

(4) Finally, let $N_G(T)$ act on T by conjugation. This gives a continuous homomorphism $N_G(T) \to \mathrm{Aut}(T) \simeq \mathrm{GL}_r(\mathbb{Z})$. Since the latter group is discrete, the connected component is in the kernel and hence $N_G(T)^\circ \subset Z_G(T)$. Since $Z_G(T) \subset N_G(T)$ is connected we also have the reverse inclusion.

□

3.4.1. Maximal tori. Fix a connected compact Lie group G.

DEFINITION 111. A *maximal torus* in G is a torus in G, maximal wrt inclusion.

LEMMA 112. *Every element $g \in G$ is contained in a torus.*

PROOF. Suppose $g = \exp(X)$ for $x \in \mathfrak{g}$. Then the closure of $\{\exp(tX)\}_{t\in\mathbb{R}}$is an abelian subgroup, hence a torus. □

COROLLARY 113. *Every element of g is contained in a maximal torus.*

Fix a maximal torus T.

COROLLARY 114. $N_G(T)^\circ = Z_G(T) = T$.

PROOF. Let $g \in Z_G(T)$ not belong to T. Then there is a torus $S \subset Z_G(T)$ such that $g \in S$. Then ST is a torus propely containing T. □

DEFINITION 115. The *Weyl group* of G is $W(G:T) \stackrel{\text{def}}{=} N_G(T)/Z_G(T) = N_G(T)/T$.

THEOREM 116. *All maximal tori of G are conjugate.*

PROOF. Let S, T be maximal tori and let $X \in \mathrm{Lie}\, S$, $Y \in \mathrm{Lie}\, T$ be generic elements (that is $\exp X, \exp Y$ are topological generators of the respective groups). Equip $\mathfrak{g} = \mathrm{Lie}\, G$ with a G-invariant inner product, and let $g \in G$ minimize

$$f(g) = \|\mathrm{Ad}(g)X - Y\|^2 .$$

Expressing f as:

$$\begin{aligned} f(g) &= \|\mathrm{Ad}(g)X\|^2 + \|Y\|^2 - 2\langle \mathrm{Ad}(g)X, Y\rangle \\ &= \|X\|^2 + \|Y\|^2 - 2\langle \mathrm{Ad}(g)X, Y\rangle \end{aligned}$$

we see that we are minimizing $\langle \mathrm{Ad}(g)X, Y\rangle$. Suppose the minimum is at g_0, and consider the derivative there. For every $Z \in \mathfrak{g}$ the derivative in the direction Z is:

$$\langle \mathrm{ad}\, Z \cdot (\mathrm{Ad}(g_0)X), Y\rangle .$$

Letting $X_0 = \mathrm{Ad}(g_0)X$ we see that

$$\begin{aligned} 0 &= \langle \mathrm{ad}\, Z \cdot (\mathrm{Ad}(g_0)X), Y\rangle \\ &= \langle [Z, X_0], Y\rangle = -\langle [X_0, Z], Y\rangle \\ &= -\langle \mathrm{ad}\, X_0 \cdot Z, Y\rangle \\ &= \langle Z, \mathrm{ad}\, X_0 \cdot Y\rangle = \langle Z, [X_0, Y]\rangle \end{aligned}$$

where we use that in every unitary representation π, $d\pi(X)$ is anti-hermitian. Since Z is arbitrary, we see that $[X_0, Y] = 0$. This means that $X_0 \in Z_{\mathfrak{g}}(Y) = \mathfrak{t}$. But since X_0 is generic for $g_0 S g_0^{-1}$ we conclude that $g_0 S g_0^{-1} = T$. □

COROLLARY 117. $T/W = G/\mathrm{Ad}(G)$.

PROOF. Fix a maximal torus T. Every $g \in G$ generates an abelian subgroup, hence contained in a maximal abelian subgroup, which is conjugate to T. It follows that every conjugacy class has a representative in T, so T/W surjects on the set of conjugacy classes. Conversely, let $t, t' \in T$ be conjugate in G. □

3.4.2. Example: three-dimensional groups. Let $G = \mathrm{SU}(2)$ act on $\mathbb{C}^2$. The action on S^3 is simply transitive, so $\mathrm{SU}(2) \simeq S^3$; in particular it is simply connected. Now $Z(\mathrm{SU}(2)) = \{\pm I\}$,so the groups it covers are $\mathrm{SU}(2)$ and its image by the adjoint representation.

LEMMA 118. *The maximal tori are the maximal subalgebras of* $\mathrm{SO}(3)$.

PROOF. Let $\mathfrak{t} = \mathrm{Span}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Then the action of $\mathfrak{t}$ on its orthogonal complement in $\mathfrak{so}(3)$ is irreducible. □

PROPOSITION 119. *Let G be a three-dimensional connected compact Lie group. Then G covers* $\mathrm{SO}(3)$.

PROOF. Consider the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$. Choosing a G-invariant inner product on $\mathfrak{g}$, the image lies in $\mathrm{O}(3)$, in fact in $\mathrm{SO}(3)$ since it is connected. We consider the Lie algebra of the image. The group

We show that the image is three-dimensional, so it equals $\mathrm{SO}(3)$ exactly. Can't be a torus (then $\mathfrak{g}/Z_{\mathfrak{g}}$ is cyclic) and can't be trivial ($\mathfrak{g}$ is noncommutative). □

3.5. Roots and weights

3.5.1. Weights. Let T be a torus. Let (π, V) be a finite-dimensional representation of T on a complex vector space. By the theory for general compact groups we have a direct sum decomposition

$$V = \oplus_{\chi \in \hat{T}} V_\chi .$$

Since T is commutative, $\hat{T} = \mathrm{Hom}_{\mathrm{cts}}\left(T, S^1\right)$ and $V_\chi = \{\underline{v} \in V \mid \pi(t)\underline{v} = \chi(t)\underline{v}\}$. We call $\left\{\chi \in \hat{T} \mid V_\chi \neq \{0\}\right\}$ the *exponential weights* of V, V_χ the *weight spaces.*

We now find an alternative parametrization of $\hat{T}$. For this let $\mathfrak{t}$ be the Lie algebra, $\exp\colon \mathfrak{t} \to T$ the exponential map. We have seen that exp is also the universal covering map of T; we write Λ for its kernel and call it the *integral lattice*.

Identify the Lie algebra of S^1 with $\mathbb{R}$ so that the exponential map is $e(z) = e^{2\pi i z}$. For a character $\chi \in \hat{T}$ write $\alpha = d\chi \in \mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ for its derivative, giving the following commutative diagram:

$$\begin{tikzcd} \mathfrak{t} \arrow[r, "\exp"] \arrow[d, "\alpha"] & T \arrow[d, "\chi"] \\ \mathbb{R} \arrow[r, "e"] & S^1 \end{tikzcd}$$

Now $\chi \circ \exp$ vanishes on Λ, and it follows that $\alpha(\Lambda) \subset \ker(e) = \mathbb{Z}$. The converse is also clear, so

CONCLUSION 120. $\chi \in \hat{T}$ iff $\alpha \in \Lambda^* = \{\nu \in \mathfrak{t}^* \mid \nu(\Lambda) \subset \mathbb{Z}\} \simeq \operatorname{Hom}(\Lambda, \mathbb{Z})$.

We call Λ^* the *weight lattice* of T, and from now on we index weight spaces with the weights $\alpha \in \Lambda^*$ rather than the corresponding exponential weights $\chi_\alpha \in \operatorname{Hom}(T, S^1)$. Explicitely given $\alpha \in \Lambda^*$ and $H \in \mathfrak{t}$ we have $\chi_\alpha(\exp H) = e^{2\pi i \alpha(H)}$.

3.5.2. Complexification. Suppose now that T acts on a *real* vector space V. Since every non-trivial character of T takes complex values, V realizes no character of T, and we consider the *complexification* $V_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} V$.

The complex conjugation operator $z \mapsto \bar{z}$ of $\mathbb{C}$ then extends to an operation $\underline{v} \mapsto \bar{\underline{v}}$ on $V_\mathbb{C}$ (fixing the image of V in $V_\mathbb{C}$), and also $\operatorname{End}_\mathbb{C}(V_\mathbb{C})$ (fixing the image of $\operatorname{End}_\mathbb{R}(V)$ there).

EXERCISE 121. A ($\mathbb{C}$-linear) subspace $W \subset V_\mathbb{C}$ is of the form $U_\mathbb{C}$ for an ($\mathbb{R}$-linear) subspace $U \subset V$ iff $W = \overline{W}$.

The T-action on V then extends to a T-action on $V_\mathbb{C}$, so we may write $V_\mathbb{C} = \bigoplus_{\alpha \in \Lambda^*} V_\alpha$. Then for any $H \in \mathfrak{t}$ and $\underline{v} \in V_\alpha$ we have

$$\pi(\exp(H)) \cdot \underline{v} = e^{2\pi i \alpha(H)} \underline{v}.$$

Taking complex conjugates it follows that

$$\pi(\exp(H)) \cdot \bar{\underline{v}} = e^{-2\pi i \alpha(H)} \bar{\underline{v}},$$

in other words that $\bar{\underline{v}} \in V_{-\alpha}$. We conclude that $\alpha \neq 0$ is a weight iff $-\alpha$ is a weight and that $\bar{V}_\alpha = V_{-\alpha}$.

3.5.3. Roots. Let G be a connected compact Lie group and fix a maximal torus $T \subset G$.

DEFINITION 122. The *rank* of G is the integer $\operatorname{rk} G = \dim T$. The *semisimple rank* of G is the rank of $G/Z(G)$, in other words the integer $\dim T - \dim Z(G)$.

DEFINITION 123. The *real roots* of G (with respect to T) are the non-zero weights of the adjoint action of T on $\mathfrak{g}$. Write $\Phi = \Phi(G : T)$ for the set of roots.

The weight space $\mathfrak{g}_0$ corresponding to the weight 0 (that is, the space of T-fixed vectors) is self-conjugate, hence is the complexification of the space of T-fixed vectors in $\mathfrak{g}$. Since $Z_G(T) = T$ we see that this is exactly $\mathfrak{t}_\mathbb{C}$ so we have

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha .$$

REMARK 124. We will now show that the structure of $\mathfrak{g}$ can be computed from this decomposition.

Let $H \in liet$, $X_\alpha \in \mathfrak{g}_\alpha$. We then have

$$\operatorname{Ad}(\exp(tH)) \cdot X_\alpha = e^{2\pi i \alpha(H)} X_\alpha .$$

Differentiating with respect to t we conclude that

$$\operatorname{ad}_H \cdot X_\alpha = 2\pi i \alpha(H) X_\alpha .$$

In other words, $\mathfrak{g}_\alpha$ is a joint eigenspace of $\{\operatorname{ad}_H\}_{H \in \mathfrak{t}}$ where the eigenvalue of H is $2\pi i \alpha(H)$.

DEFINITION 125. Given a real root α, the map $H \mapsto 2\pi i\alpha(H)$ will be called the associated *complex root*. We denote both by α, but it should be clear from context which is intended. Note that the real root is an element of $\mathfrak{t}_{\mathbb{R}}^*$ while the latter is a purely imaginary element of $\mathfrak{t}_{\mathbb{C}}^*$. Generaly the real roots are useful when studying representation theory and the "root system". The complex roots are useful when studying structure theory, that is in computing commutators in $\mathfrak{g}$. Recall that we also have an associated *exponential root* $\chi_\alpha \colon T \to S^1$ such that $\mathrm{Ad}_t \cdot X_\alpha = \chi_\alpha(t) X_\alpha$ whenever $t \in T$, $X_\alpha \in \mathfrak{g}_\alpha$.

LEMMA 126. *For* $\alpha, \beta \in \Lambda^*$, $\left[\mathfrak{g}_\alpha, \mathfrak{g}_\beta\right] \subset \mathfrak{g}_{\alpha+\beta}$.

PROOF. Let $H \in liet$, $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$. Then by the Jacobi identity (writing α for the *complex* root)

$$\begin{aligned}
\left[H, \left[X_\alpha, X_\beta\right]\right] &= -\left[X_\alpha, \left[X_\beta, H\right]\right] - \left[X_\beta, \left[H, X_\alpha\right]\right] \\
&= -\left[X_\alpha, -\beta(H) X_\alpha\right] - \left[X_\beta, \alpha(H) X_\alpha\right] \\
&= \left(\beta(H) + \alpha(H)\right)\left[X_\alpha, X_\beta\right] \\
&= \left((\alpha+\beta)(H)\right)\left[X_\alpha, X_\beta\right].
\end{aligned}$$

□

We are now ready to begin studying structure theory in earnest The following argument is taken from [Brocker–tom Dieck, Prop xxxx]

THEOREM 127. *If* $\mathrm{rk}\, G = 1$ *then* G *is either* $\mathrm{SO}(3)$ *or* $\mathrm{SU}(2)$.

PROOF. We begin with two preliminary observations

(1) Given $\beta \in \Phi$ let $X_\beta \in \mathfrak{g}_\beta$. Then $X_{-\beta} = \bar{X}_\beta \in \mathfrak{g}_{-\beta}$ and we may consider $H_\beta = \left[X_\beta, X_{-\beta}\right]$. If H_β were zero $\mathrm{Span}\left\{X_\beta, X_{-\beta}\right\} \subset \mathfrak{g}_{\mathbb{C}}$ would be a two-dimensional commutative subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Since this subspace is stable by complex conjugation it would be the complexification of a two-dimensional commutative subalgebra of *lieg*, and such subalgebras don't exist when $\mathrm{rk}\, G = 1$. It follows that $H_\beta \neq 0$ in such circumstances. We also note that $\bar{H}_\beta = \left[\bar{X}_\beta, \bar{X}_{-\beta}\right] = \left[X_{-\beta}, X_\beta\right] = -H_\beta$. It follows that $H_\beta \in i\mathfrak{t}_{\mathbb{R}}$, and that $iH_\beta \in \mathfrak{t}$.
(2) Fix a non-zero $H \in \mathfrak{t}$. Sine $\mathfrak{t}$ is one-dimensional, every real root α is determined by the non-zero real number $\alpha(H)$, and we *order* the roots by these numbers.

Now let β be the smallest positive root, choose X_β and $X_{-\beta}$ as above and let

$$V = \mathbb{C} X_{-\beta} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha .$$

We then have:

(1) V is ad_{X_β}-invariant, since $\mathrm{ad}_{X_\beta} \cdot X_{-\beta} \subset \mathfrak{g}_0$, and for $\alpha \geq 0$ $\mathrm{ad}_{X_\beta} \cdot \mathfrak{g}_\alpha \subset \mathfrak{g}_{\alpha+\beta}$ and $\alpha + \beta \geq 0$.
(2) V is $\mathrm{ad}_{X_{-\beta}}$-invariant, since $\mathrm{ad}_{X_{-\beta}} \cdot X_{-\beta} = 0$, $\mathrm{ad}_{X_{-\beta}} \cdot \mathfrak{t}_{\mathbb{C}} \subset \mathbb{C} X_{-\beta}$ and for any $\alpha > 0$ we have $\alpha \geq \beta$ so $\mathrm{ad}_{X_{-\beta}} \cdot \mathfrak{g}_\alpha \subset \mathfrak{g}_{\alpha-\beta}$ with $\alpha - \beta \geq 0$.

Now let $H_\beta = \left[X_\beta, X_{-\beta}\right]$ as above. Since the adjoint representation is a Lie algebra representation (Corollary 94), $\mathrm{ad}_{H_\beta} = \left[\mathrm{ad}_{X_\beta}, \mathrm{ad}_{X_{-\beta}}\right]$ so V is also stable by ad_{H_α}. since ad_{H_α} is a commutator it follows that $\mathrm{Tr}_{\mathbb{C}}\left(\mathrm{ad}_{H_\beta}\left|V\right.\right) = 0$. On the other hand, we can compute this trace via the eigenspace decomposition:

$$\mathrm{Tr}_{\mathbb{C}}\left(\mathrm{ad}_{H_\beta}\left|V\right.\right) = 2\pi i\beta(H_\beta) + 0 + \sum_{\alpha>0} \dim_{\mathbb{C}} \mathfrak{g}_\alpha \cdot 2\pi i\alpha(H_\beta)\,.$$

Rearranging the terms we conclude that

$$\left(\dim_{\mathbb{C}} \mathfrak{g}_\beta - 1\right)\beta\left(iH_\beta\right) + \sum_{\alpha>\beta} \dim_{\mathbb{C}} \mathfrak{g}_\alpha \cdot \alpha(iH_\beta) = 0\,.$$

Now $iH_\beta \in \mathfrak{t}$ is a non-zero multiple of H. In particular either all the numbers $\beta(iH_\beta), \alpha(iH_\beta)$ are all positive or they are all negative. Also, the coefficients $\left(\dim_{\mathbb{C}} \mathfrak{g}_\beta - 1\right), \dim_{\mathbb{C}} \mathfrak{g}_\alpha$ are all non-negative. It follows that $\dim_{\mathbb{C}} \mathfrak{g}_\beta = \dim_{\mathbb{C}} \mathfrak{g}_{-\beta} = 1$ and that $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 0$ if $\alpha > \beta$, in other words that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{-\beta} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_\beta$ is three dimensional. □

3.5.4. The algebraic Weyl group. Continuing with our general group G and maximal torus T, let $\alpha \in \Phi$ and let $\mathfrak{u}_\alpha = \ker(\alpha)$, a codimension-1 subspace of $\mathfrak{t}$, $G_\alpha = Z_G(\mathfrak{u}_\alpha)$.

LEMMA 128. *$\mathfrak{u}_\alpha$ is the Lie algebra of the kernel of the exponential root χ_α. In particular,* $\exp(\mathfrak{u}_\alpha)$ *is a closed subgroup of T of codimension* 1.

REMARK 129. That kernel need not be connected (for example, the kernel of the root of $\mathrm{SU}(2)$ consists of the disconnected centre). We will later see that this kernel has at most two connected components.

PROPOSITION 130. *G_α is a connected subgroup of semisimple rank* 1. *Moreover:*

(1) $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = \dim_{\mathbb{C}} \mathfrak{g}_{-\alpha} = 1$ *and $\pm\alpha$ are the only roots proportional to α.*
(2) $W(G_\alpha : T) \simeq C_2$.
(3) *Let $s_\alpha \in W\left(G_\alpha : T\right) \subset W(G : T)$ be the non-trivial element. Then $s_\alpha \in \mathrm{GL}(\mathfrak{t})$ is a reflection in the hyperplane u_α.*

PROOF. G_α centralizes the Lie algebra of a torus, so by Lemma 110 it is connected. Since $T \supset \exp(\mathfrak{u}_\alpha)$ is commutative, we see that $T \subset G_\alpha$ so that T is a maximal torus there as well. By construction, $\mathfrak{u}_\alpha \subset Z_{\mathrm{Lie}\,G_\alpha}$ so the semisimple rank is at most 1. It is not zero then G_α is non-commutative: its lie algebra contains both $\mathfrak{t}$ and $\Re\left(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\right)$, and these subspace do not commute.

Set $\bar{G}_\alpha = G_\alpha / \operatorname{Ker}\chi_\alpha$, and let $\bar{T} = T/\operatorname{Ker}(\chi_\alpha)$, a maximal torus there. This is a group of rank 1, hence isomorphic to one of $\mathrm{SU}(2), \mathrm{SO}(3)$.

(1) Let β be a root proportional to α. Then $\pm\beta(H) = 0$ for any $H \in \mathfrak{u}_\alpha$ and it follows that $\Re\left(\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta}\right) \subset \mathrm{Lie}\,G_\alpha$ and hence that $\mathfrak{g}_\beta \subset \mathrm{Lie}\,\mathbb{C}G_\alpha$. The direct sum over all these subspaces is disjoint from $\mathfrak{u}_\alpha$ so they all inject into $\mathrm{Lie}\,\mathbb{C}G_\alpha/\mathfrak{u}_{\alpha\mathbb{C}}$. Being the complexified Lie algebra of $\bar{G}_\alpha$ it is three-dimensional and it follows that $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = \dim_{\mathbb{C}} \mathfrak{g}_{-\alpha} = 1$ and that there are no other roots proportional to α.
(2) If $g \in G_\alpha$ normalizes T then its image in $\bar{G}_\alpha$ normalizes its maximal torus $\bar{T}$. Conversely, if the image of g normalizes $\bar{T}$ then for any $t \in T$ we have $gtg^{-1} \in T\operatorname{Ker}(\chi_\alpha) = T$ so g normalizes T. It follows that the quotient map induces an isomorphism of the Weyl groups $W\left(G_\alpha; T\right) \simeq W\left(\bar{G}_\alpha : \bar{T}\right) \simeq C_2$.

(3) Since $\mathfrak{u}_\alpha$ is central in $\operatorname{Lie} G_\alpha$ it is fixed by any element of G_α. The non-trivial element of $W\left(\bar{G}_\alpha : \bar{T}\right)$ acts by inversion on $\bar{T}$, so s_α acts by inversion on $\mathfrak{t}/\mathfrak{u}_\alpha$, that is by reflection in $\mathfrak{u}_\alpha$ on $\mathfrak{t}$.

$\square$

REMARK 131. We call a root *reduced* if it is not a multiple of another root, and we see that here every root is reduced.

Since $N_{G_\alpha}(T) \subset N_G(T)$ we can think of $s_\alpha \in N_{G_\alpha}(T)/T$ as an element of $W = N_G(T)/T$. This element is a reflection on $\mathfrak{t}$ fixing $\mathfrak{u}_\alpha$. Having equipped $\mathfrak{g}$ with an inner product, the Weyl group acts by isometries on $\mathfrak{t}$ so s_α must be the orthogonal reflection in $\mathfrak{u}_\alpha$. We note that W also acts on the dual space $\mathfrak{t}^*$ fixing the dual lattice Λ^* and the roots Φ and that $s_\alpha(\alpha) = -\alpha$.

DEFINITION 132. Call s_α the *root reflection* associated to the root α. The subgroup of the Weyl group generated by the root reflections will be called the *algebraic Weyl group.*

COROLLARY 133. *Let* $\mathfrak{z} = Z(\mathfrak{g})$ *be the Lie algebra of the centre of G and let* $V = \{\nu \in \mathfrak{t}^* \mid \nu(\mathfrak{z}) = 0\} = (\mathfrak{t}/\mathfrak{z})^*$. *Then*$(V,\Phi)$ *is a* root system*, in that it has the following properties:*

(1) $\Phi \subset V$ *is a finite set not containing* $\{0\}$.
(2) $\operatorname{Span}_{\mathbb{R}} \Phi = V$.
(3) *For every* $\alpha \in \Phi$*, the reflection* s_α *in the hyperplane perpendicular to* α *preserves* Φ *setwise.*

EXAMPLE 134. Let $G = \mathrm{SU}(3)$. Let $T = \{\operatorname{diag}(e(i\theta_1), e(i\theta_2), e(i\theta_3)) \mid \theta_1 + \theta_2 + \theta_3 = 0\}$. This is a torus (isomorphic to $(S^1)^2$). To see that it is maximal and compute its Weyl group, restrict the standard representation of $\mathrm{SU}(3)$ on $\mathbb{C}^3$ to T. The coordinate axes are exactly the irreducible subrepresentations and they are non-isomorphic (each is one copy of a different character). It follows that every $w \in N_G(T)$ must permute these subspaces and every $t \in Z_G(T)$ must act on each subspace separately. But these subspaces are irreducible, so each $t \in Z_G(T)$ must be diagonal, and hence an element of t. It follows that $T = Z_G(T)$ so it is a maximal torus, that $N_G(T)$ is the group of signed permutations, and that $W(G:T) = N_G(T)/T \simeq S_3$.

Differentiaing the definition $G = \left\{g \in \mathrm{SL}_3(\mathbb{C}) \mid g^\dagger g = \mathrm{Id}\right\}$ we see that $\mathfrak{g} = \left\{X \in_3 \mathbb{C} \mid \mathbb{X}^\dagger + \mathbb{X} = \not{\vdash}\right\}$ that is the set of anti-Hermitian matrices of trace zero. Since every $Y \in_3 \mathbb{C}$ can be uniquely written in the form

$$Y = \frac{Y + Y^\dagger}{2} + \frac{Y - Y^\dagger}{2} = \frac{Y - Y^\dagger}{2} + i\frac{Y + Y^\dagger}{2i} \in \mathfrak{g} \oplus i\mathfrak{g}$$

we see that $\mathfrak{g}_{\mathbb{C}} \simeq_3 \mathbb{C}$. It is also clear that $\mathfrak{t} = \{i \operatorname{diag}(\theta_1, \theta_2, \theta_3) \mid \theta_1 + \theta_2 + \theta_3 = 0\}$.

Now for $i \neq j$ let $E^{ij} \in_3 \mathbb{C} \subset \mathbb{M}_{\not{\mathbb{K}}}(\mathbb{C})$ be the matrix with zeroes everywhere except that $\left(E^{ij}\right)_{ij} = 1$. Then for $H = i \operatorname{diag}(\theta_1, \theta_2, \theta_3)$ we have $\operatorname{ad}_H \cdot E^{ij} = i(\theta_i - \theta_j)E^{ij}$ so the roots of G are the maps $e_{ij}(H) = \theta_i - \theta_j$.

To find the Weyl chamber we note that the Frobenius, or Hilbert–Schmidt norm on $M_3(\mathbb{C})$ is $\mathrm{U}(3)$-invariant. In terms of this norm (and removing the factor of i) an orthonormal basis of $\mathfrak{t}$ is given by $\frac{1}{\sqrt{6}} \operatorname{diag}(1,1,-2), \frac{1}{\sqrt{2}} \operatorname{diag}(1,-1,0)$. Now for

$$H = \frac{x}{\sqrt{6}} \operatorname{diag}(1,1,-2) + \frac{y}{\sqrt{2}} \operatorname{diag}(1,-1,0)$$

we have

$$
\begin{aligned}
e_{12}(H) &= \sqrt{2}y \\
e_{23}(H) &= \frac{\sqrt{3}}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \\
e_{13}(H) &= \frac{\sqrt{3}}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y .
\end{aligned}
$$

In the coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$ we therefore have:

$$\mathfrak{u}_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\perp}, \mathfrak{u}_{23} = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}^{\perp}, \mathfrak{u}_{13} = \begin{pmatrix} \sqrt{3}/2 \\ +1/2 \end{pmatrix}^{\perp}.$$

These three lines are the lines at slopes $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ throug the origin, dividing $\mathbb{R}^2$ into six identical sectors. We call these sectors *Weyl chambers*, the lines *walls*, and note that S_3 (which has order 6) acts on the six chambers simply transitively.

EXERCISE 135. Do the same for $\mathrm{SU}(n)$, $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(n)$.

3.5.5. Weyl chambers. The complement of hyperplane $\mathfrak{u}_\alpha$ consists of two half-spaces: the sets $\{H \in \mathfrak{t} \mid \alpha(H) > 0\}$ and $\{H \in \mathfrak{t} \mid \alpha(H) < 0\}$. It follows that the connected components of

$$\mathfrak{t} \setminus \bigcup_{\alpha \in \Phi} \mathfrak{u}_\alpha$$

are interections of half-spaces, hence convex cones.

DEFINITION 136. These connected components are called the (open) *Weyl chambers* in $\mathfrak{t}$. We call $\mathfrak{u}_\alpha$ a *wall* of the chamber C if $\dim\left(\mathfrak{u}_\alpha \cap \bar{C}\right) = \operatorname{rk} G - 1$. More generally, a (codimension-k-) *facet* of the Weyl chamber C is any non-empty set of the form $F = \left(\mathfrak{u}_{\alpha_1} \cap \cdots \mathfrak{u}_{\alpha_k} \cap \bar{C}\right)^\circ$ where the interior is taken as a subset of the vector space $\mathfrak{u}_{\alpha_1} \cap \cdots \mathfrak{u}_{\alpha_k}$. We note that the closure $\bar{C}$ is the disjoint union of the facets of C (where C itself is the unique facet of codimension zero).

REMARK 137. Note that we are studying the Weyl chambers in *liet*, rather than the Weyl chambers in $\mathfrak{t}^*$ where the root system lies.

Given a chamber C, let Δ be the set of roots α such that $\mathfrak{u}_\alpha$ is a wall of C and such that α is positive on C (note that $\mathfrak{u}_\alpha = \mathfrak{u}_{-\alpha}$ and that exactly one of $\alpha, -\alpha$ is positive on C).

FACT 138. *The chamber is exactly the set bounded by the walls:* $C = \{H \in \mathfrak{t} \mid \forall \alpha \in \Delta : \alpha(H) > 0\}$.

OBSERVATION 139.

(1) *The Weyl group acts on G by automorphisms while fixing T. It therefore permutes the roots, hence their kernels, and hence the Weyl chambers.*
(2) *It is clear that there is a bijection between Weyl chambers and (satisfiable)* notions of positivity *(choices of sign for all the $\alpha(H)$). The facets are determined by having some roots positive, some negative, and some vanishing.*

LEMMA 140. *The group $W' = \left\langle \{s_\alpha\}_{\alpha \in \Delta} \right\rangle$ acts transitively on the set of Weyl chambers.*

PROOF. Fix $x \in C$; let C' be any other chamber and let $y \in C'$. Note that (being equivalence classes for an equivalence relation) if two chambers intersect they are equal, to it suffices to show that $wy \in C$ for some $w \in W'$. For this choose w such that $\|wy - x\|$ is minimal. If $wy \notin C$ then by Fact 138 above, there is a wall $\mathfrak{u}_\alpha$ such that x, wy are on opposite sides of α. Decomposing x, wy into their components along and perpendicular to $\mathfrak{u}_\alpha$ it is then clear that

$$\|s_\alpha(wy) - x\| < \|wy - x\| ,$$

which is a contradiction since $s_\alpha w \in W'$. $\square$

LEMMA 141. *The group W acts simply transitively on the chambers.*

PROOF. We already know the action is transitive. Suppose $w \in N_G(T)$ stabilizes the chamber C. Since W is finite, w has finite order as an automorphism of T so averaging over a w-orbit shows that w fixes some $x \in C$ (recall that C is convex). Think of x as an element $H \in liet$, we have that $\mathrm{Ad}_w \cdot H = H$, that is $w \in Z_G(H)$.

On the other hand, since $H \in C$, $\alpha(H) \neq 0$ for all $\alpha \in \Phi$. It follows that H acts non-trivially in every root space so $Z_{\mathfrak{g}_\mathbb{C}}(H) = \mathfrak{t}_\mathbb{C}$ and hence $Z_\mathfrak{g}(H) = liet$. Now $Z_G(H)$ is connected (this is true for all $H \in lieg$); its lie Algebra being $Z_\mathfrak{g}(H)$ we conclude that $Z_G(H) = T$ and hence that $w \in T$. It follows that the image of w in $W = N_G(T)/T$ is trivial. $\square$

COROLLARY 142. *$W' = W$, that is the algebraic and analytic Weyl groups coincide.*

PROOF. Let $w \in W$. By the transitivity of W' there is $w' \in W'$ such that $w \cdot C = w' \cdot C$. By the simplicity of the action we conclude $w = w' \in W'$. $\square$

3.5.6. Geometry of the roots. The linear map $s_\alpha - \mathrm{Id}_\mathfrak{t}$ is non-zero but vanishes on $\mathfrak{u}_\alpha$. It therefore has rank 1, and factors through α. We conclude that there is a unique $\check{\alpha} \in \mathfrak{t}$ such that

$$s_\alpha(x) = x - \alpha(x)\check{\alpha} .$$

The dual action on $\mathfrak{t}^*$ is then

$$s_\alpha(\nu) = \nu - \nu(\check{\alpha})\alpha$$

and since $s_\alpha(\alpha) = -\alpha$ we have $\alpha(\check{\alpha}) = 2$.

DEFINITION 143. Call $\check{\alpha}$ the *coroot* associated to α and write $\check{\Phi}$ for the set of coroots.

REMARK 144. If $\alpha + \beta$ is a root it need not be the case that $\alpha \check{+} \beta = \check{\alpha} + \check{\beta}$. In particular, a root system and its dual need not be isomorphic.

EXERCISE 145. $(\mathfrak{t}/\mathfrak{z}, \check{\Phi})$ is a root system, the *dual root system*.

LEMMA 146. *Coroots are integral, that is $\check{\alpha} \in \Lambda = \mathrm{Ker}(\exp\restriction_\mathfrak{t})$.*

PROOF. The element $\frac{1}{2}\check{\alpha}$ has $\exp\left(2\pi i \alpha(\frac{1}{2}\check{\alpha})\right) = \exp(2\pi i) = 1$ since $\alpha(\check{\alpha}) = 2$. In other words, $\exp\left(\frac{1}{2}\check{\alpha}\right)$ lies in the kernel the exponential root χ_α and hence is $\mathrm{Ad}(s_\alpha)$-stable. On the other hand, $s_\alpha(\check{\alpha}) = -\check{\alpha}$ so $\mathrm{Ad}(s_\alpha)\exp\left(\frac{1}{2}\check{\alpha}\right) = \exp\left(-\frac{1}{2}\check{\alpha}\right)$. It follows that

$$\exp\left(\frac{1}{2}\check{\alpha}\right) = \exp\left(-\frac{1}{2}\check{\alpha}\right) ,$$

that is $\exp(\check{\alpha}) = 1$ and $\check{\alpha} \in \Lambda$. $\square$

COROLLARY 147. *For any $\alpha, \beta \in \Phi$ we have $n_{\alpha\beta} = \beta(\check{\alpha}) \in \mathbb{Z}$.*

DEFINITION 148. The $n_{\alpha\beta}$ are called the *Cartan numbers* of $\mathfrak{g}$. Note that $s_\alpha(\beta) = \beta - n_{\alpha\beta}\alpha$.

DEFINITION 149. The *coroot lattice* is the subgroup $\Gamma < \Lambda$ generated by the coroots.

FACT 150. $\Lambda/\Gamma \simeq \pi_1(G)$.

COROLLARY 151. *$\tilde{G}$ is compact iff $\pi_1(G)$ is finite iff $\check{\Phi}$ spans $\mathfrak{t}$ iff $\mathfrak{z} = 0$ iff $Z(G)$ is finite. In each of those equivalent cases we say that G is* semisimple.

FACT 152. *G is semisimple iff its lie algebra is the direct sum of nonabelian simple lie algebras, iff G is the almost direct product of nonabelian quasisimple groups.*

Recall that we have equipped $\mathfrak{g}$ with an invariant inner product. This also endows $\mathfrak{t}^*$ with an inner product and then

$$s_\alpha(\nu) = \nu - 2\frac{\langle \nu, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha$$

so if we identify *liet*, $\mathfrak{t}^*$ using this inner product the element $\check{\alpha}$ is identified with $\check{\alpha}^* = \frac{2\alpha}{\langle \alpha,\alpha\rangle} \in \mathfrak{t}^*$. Now $n_{\alpha\beta} = \beta(\check{\alpha}) = \langle \beta, \check{\alpha}^*\rangle = 2\frac{\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle}$. It follows that

$$n_{\alpha\beta}n_{\beta\alpha} = 4\frac{\langle \alpha, \beta\rangle^2}{\langle \alpha, \alpha\rangle\langle \beta, \beta\rangle} \le 4$$

by Cauchy–Schwartz, with equality iff α, β are proportional. Since the two Cartan numbers are integers, each is zero iff $\alpha \perp \beta$, and if both are non-zero their product is positive, we see that (up to exchanging α, β) if α, β are not proportional, the pair $(n_{\alpha\beta}, n_{\beta\alpha})$ must be one of the seven possibilities:

$$(0,0), \pm(1,1), \pm(1,2), \pm(1,3).$$

In each case the pair $(n_{\alpha\beta}, n_{\beta\alpha})$ determines the angle between the roots and (if they are not orthogonal) the ratio of their lengths.

COROLLARY 153. *Let α, β be non-proportional and suppose that that $n_{\alpha\beta} > 0$ (equivalently that $\langle \alpha, \beta\rangle > 0$). Then $\alpha - \beta \in \Phi$.*

PROOF. If $n_{\alpha\beta} > 0$ then either $n_{\beta\alpha} = 1$, at which point $s_\beta(\alpha) = \alpha - \beta \in \Phi$, or $n_{\alpha\beta} = 1$, at which point $s_\beta(\alpha) = \beta - \alpha \in \Phi$ and then $\alpha - \beta \in \Phi$ as well. □

3.5.7. Simple roots. Fix a Weyl chamber C, giving a *notion of positivity*: call $\alpha \in \Phi$ *positive* if it is positive on C, negative otherwise, and write Φ^+, Φ^- for the sets of positive and negative roots. Since roots have constant sign on C it suffices to evaluate them at a fixed $H \in C$.

DEFINITION 154. Call $\alpha \in \Phi^+$ *simple* if it is not a sum of positive roots, and let Δ be the set of simple roots.

REMARK 155. This clearly depends on the choice of C. More on that anon.

LEMMA 156. *Every positive root is a positive sum of simple roots.*

PROOF. Let α be a counterexample with $\alpha(H)$ minimal. Then α is not a simple root, so $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Phi^+$. But then $\beta(H) + \gamma(H) = \alpha(H)$ shows that $\beta(H), \gamma(H) < \alpha(H)$ so they are sums of positive roots and we have a contradiction. □

PROPOSITION 157. *$\Delta \subset \mathfrak{t}^*$ is linearly independent.*

PROOF. Let $\alpha, \beta \in \Delta$ be distinct. If the angle between them was acute ($\langle \alpha, \beta \rangle > 0$) then by Corollary 153 one of $\alpha - \beta, \beta - \alpha$ would be a positive root and this would make either α or β decomposable. It follows that $\langle \alpha, \beta \rangle \leq 0$ for each pair. they are also all contained in the half-plane $\{\nu \mid \nu(H) > 0\}$. We show these two hypotheses suffice to make a set of vectors independent. Indeed, suppose we have a linear dependence in Δ. We then have disjoint non-empty $A, B \subset \Delta$ and *positive* coefficients $\{a_\alpha\}_{\alpha \in A}, \{b_\beta\}_{\beta \in B}$ such that

$$\sum_{\alpha \in A} a_\alpha \cdot \alpha = \sum_{\beta \in B} b_\beta \cdot \beta \,.$$

Call this vector ν. Then

$$0 \leq \langle \nu, \nu \rangle = \sum_{\alpha, \beta} a_\alpha b_\beta \langle \alpha, \beta \rangle \leq 0$$

and it follows that $\nu = 0$. We therefore have

$$0 = \nu(H) = \sum_{\alpha \in A} a_\alpha \cdot \alpha(H) > 0 \,,$$

a contradiction. □

LEMMA 158. Δ *spans* $(\mathfrak{g}/\mathfrak{z})^*$.

PROOF. Every simple root vanishes on $\mathfrak{z}$, so the same holds for every element of the span. Conversely, the span contains Φ; it follows the common kernel of the span is exactly $\mathfrak{z}$ so the span is exactly $(\mathfrak{g}/\mathfrak{z})^*$. □

COROLLARY 159. $\#\Delta$ *is the semisimple rank.*

LEMMA 160. $\{\mathfrak{u}_\alpha\}_{\alpha \in \Delta}$ *are the walls of* C.

PROOF. $\{H \mid \forall \alpha \in \Delta : \alpha(H) > 0\} = \{H \mid \forall \alpha \in \Phi^+ : \alpha(H) > 0\} = C$. Since Δ are independent they are all walls. □

DEFINITION 161. A *system of simple roots* (or *simple system*) is a subset $\Delta \subset \Phi$ such that every root is either the sum of elements of Δ or the negative of such a sum.

COROLLARY 162. *Every system of simple roots is the set of walls of a Weyl chamber, we have a bijection between systems of simple roots, notions of positivity, and Weyl chambers, and the Weyl group acts transitively on simple systems. In particular, every root belongs to a simple system.*

CHAPTER 4

Semisimple Lie groups

CHAPTER 5

Representation theory of real groups

APPENDIX A

Functional Analysis

In this appendix we review the basics of topological vector spaces. References include TVS.

A.1. Topological vector spaces

Let K be a non-discrete complete valued field

DEFINITION 163. A *topological vector space* is a vector space V over K equipped with a topology so that $(V,+)$ is a topological group and such that scalar multiplication is a continuous map $\cdot\colon K\times V\to V$.

PROPOSITION 164. *A finite-dimensional K-vector space has a unique topology making it into a TVS. In particular, if V,W are TVS with V finite-dimensional then* $\mathrm{Hom}_K(V,W)=\mathrm{Hom}_{cts}(V,W)$ *and if $V\subset W$ then V is closed and complete. If K is locally compact then a TVS over K is locally compact iff it is finite-dimensional.*

DEFINITION 165. *Fix a TVS V. Call $C\subset V$:*

1. *Balanced*, if $\alpha\underline{v}\in C$ for all $x\in C$, $|\alpha|\le 1$
2. *Absorbing*, if $\cup_{t>0}tC=V$ (that is, for all $\underline{v}\in V$ there are $\underline{u}\in C$ and $t>0$ such that $t\underline{u}=\underline{v}$.
3. *Bounded*, if for every open neighbourhood $W\ni\underline{0}$ there is $t>0$ such that $C\subset tW$.
4. *Totally bounded,* if for every open neighbourhood $W\ni\underline{0}$ there is a finite set $\{\underline{u}_i\}_{i=1}^n\subset V$ such that $C\subset\cup_i(\underline{v}_i+W)$.

LEMMA 166. *Every finite subset of a TVS is bounded.*

LEMMA 167. *Every TVS has a basis neighbourhoods of $\underline{0}$ which are balanced.*

DEFINITION 168. A net $\{x_\alpha\}_{\alpha\in D}\subset V$ is called a *Cauchy net* if for every neighbourhood W of 0 there is $\delta\in D$ such that if $\alpha,\beta\ge\delta$ then $x_\alpha-x_\beta\in W$. $X\subset V$ is *complete* if every Cauchy net in X converges to a limit in X. V is *quasi-complete* if every closed bounded subset of X is complete.

LEMMA 169. *In a quasi-complete TVS every totally bounded subset is relatively compact.*

ASSUMPTION 170. $K=\mathbb{R}$ *or* $\mathbb{C}$.

DEFINITION 171. *Fix a TVS V. Call $C\subset V$ convex*, if $t\underline{u}+(1-t)\underline{v}\in C$ for all $\underline{u},\underline{v}\in C$, $t\in[0,1]$. Call V locally convex *if any neighbourhood of* 0 *contains a convex neighbourhood of zero.*

PROPOSITION 172. *A TVS is locally convex iff its topology is determined by a family of semi-norms.*

LEMMA 173. *Let V be locally convex, $C\subset V$ be totally bounded. Then the convex hull and balanced convex hull of C are also totally bounded.*

COROLLARY 174. *Let V be locally convex and quasi-complete and let $C\subset V$ be compact. Then the closed convex hull of C is compact.*

DEFINITION 175. The *continuous dual* of V is $V' \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathrm{cts}}(V,K)$.

THEOREM 176 (Hahn–Banach). *Let V be locally convex, $U \subset E$ a subspace, $f \in U'$. Then f has a continuous linear extension to V. In particular, V' separates the points of V.*

A.2. Quasicomplete locally convex TVS

[based on Casseleman, Garrett]

PROPOSITION 177. *An inverse limit of quasi-complete spaces is quasi-complete. The direct product of a family of quasi-complete space is quasi-complete. The weak-* dual of a Banach space is quasi-complete.*

Let V be a locally convex TVS.

DEFINITION 178. Let Ω be a measureable space.

(1) Call $f\colon \Omega \to V$ *weakly measurable* if $\varphi \circ f\colon \Omega \to K$ is measurable for each $\varphi \in V'$. Let
(2) Let μ be a measurae on Ω and let $f\colon \Omega \to V$ be weakly measurable. Call $\underline{v} \in V$ the *Gelfand–Pettis integral* of f (and write $\underline{v} = \int f \,\mathrm{d}\mu$) if for every $\varphi \in V'$ $\varphi \circ f$ is μ-integrable and we have
$$\varphi(\underline{v}) = \int_\Omega \varphi \circ f \,\mathrm{d}\mu\,.$$

REMARK 179. Note that the integral clearly exists as an element of V''; the question is about existence as an element of V. Since V' separates the points, it is also clear that the integral (if it exists) is unique.

THEOREM 180. *Let V be quasi-complete, let Ω be compact, μ a Radon measure, and let $f\colon \Omega \to V$ be continuous. Then $\int f \,\mathrm{d}\mu$ exists.*

PROOF. Wlog μ is a probability measure. In that case we also show $\int f \,\mathrm{d}\mu$ lies in the closed convex hull of $f(\Omega)$.

LEMMA 181. *If V is finite-dimensional then $\int f \,\mathrm{d}\mu$ exists and lies in the convex hull of $f(\Omega)$.*

Write C for the closed convex hull of $f(\Omega)$. For every finite $\mathcal{F} \subset V'$ consider the continuous linear map $\mathcal{F}\colon V \to K^{\mathcal{F}}$ given by $\underline{v} \mapsto (\varphi(\underline{v}))_{\varphi\in\mathcal{F}}$. It maps C continuously onto the convex hull of the image of $\mathcal{F} \circ f$. Now $\int_\Omega (\mathcal{F} \circ f)\,\mathrm{d}\mu$ exists in that convex hull, and we obtain a non-empty closed convex subset
$$C_{\mathcal{F}} = \left\{ \underline{v} \in C \mid \mathcal{F}(\underline{v}) = \int_\Omega (\mathcal{F} \circ f)\,\mathrm{d}\mu \right\}.$$
Since $\bigcap_{i=1}^r C_{\mathcal{F}_i} = C_{\bigcup_i \mathcal{F}_i}$ we see that this family has the finite intersection property, and it follows that
$$\bigcap_{\mathcal{F}} C_{\mathcal{F}}$$
is non-empty. The (necessarily unique) point there is the desired integral. □

A.3. Integration

A.4. Spectral theory and compact operators

A.5. Trace-class operators and the simple trace formula