Math 322, lecture 23, 28/11/2017

Annocement: No need to score 80% on parts of the final to pass the course.

Question; HW: correspondence thm was stated:

$f \in \mathrm{Hom}(G,H) \rightsquigarrow$ get bijection $\left\{\begin{array}{c}\text{subgps of } G \\ \text{containing} \\ K = \mathrm{Ker}(f)\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{subgps of} \\ \text{image of } f\end{array}\right\}$

Books: $K \lhd G$, $q: G \to G/K$ quotient map

$\rightsquigarrow$ get bijection $\left\{\begin{array}{c}\text{subgps of } G \\ \text{containing} \\ K\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{subgps of} \\ G/K\end{array}\right\}$

Are these same?

($\Downarrow$) $\mathrm{Ker}(q) = K$, $\mathrm{im}(q) = G/K$

($\Uparrow$) by first isom thm, write $f = \bar{f} \circ q$

q = quotient map $G \to G/K$, $\bar{f}: G/K \to \mathrm{im}(f)$ is an isom

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Announcement: Thursday class = review, i.e. you bring problems

Extra office hours: in tomorrow 10:00-12:30

Friday ~~[illegible]~~ 10:00-12:00

Monday most of the day.

Def: (Last time) $G' = G^{(1)} \overset{\text{def}}{=} [G, G]$

define inductively $G^{(i+1)} = (G^{(i)})' = [G^{(i)}, G^{(i)}]$.

Saw: $G' \triangleleft G$ so $G = G^{(0)} \triangleright G^{(1)} \triangleright G^{(2)} \triangleright \cdots$ is a normal series

Also, G/G' is abelian. (if $a,b \in G$, $[a,b] \in G'$ so a,b commute mod G')

$\Rightarrow$ if $G^{(r)} = \{e\}$, then $G^{(0)} \triangleright G^{(1)} \triangleright \cdots \triangleright G^{(r)}$ is a normal series with abelian quotients

Thm: G is solvable iff $G^{(r)} = \{e\}$ for some r.

Pf: Let $G_0 = G_0 \triangleright G_1 \triangleright G_2 \cdots$ be a normal series with abelian quotients

Then $G_0/G_1 = G/G_1$ is abelian, so if $a,b \in G$, $[a,b] \in G_1$

so $G_1 \supset \{[a,b] : a,b \in G\}$ so $G_1 \supset [G,G] = G^{(1)}$.

suppose by induction that $G_i \supset G^{(i)}$.

Then G_i/G_{i+1} abelian, so if $a,b \in G_i$, $[a,b] \in G_{i+1}$

so if $a,b \in G^{(i)}$, $[a,b] \in G_{i+1}$, so $G_{i+1} \supset \langle \{[a,b] : a,b \in G^{(i)}\} \rangle$.

i.e. $G_{i+1} \supset G^{(i+1)}$.

Now $G_r = \{e\}$, so $G^{(r)} = \{e\}$

Similarly, let $f: G \to G/N$ be the quotient map,
set $\overline{G_i} = f(G_i)$. Then $\overline{G_0} = G/N \supset \overline{G_1} \supset \cdots \overline{G_r} = \{e\}$

If $\bar{g} \in \overline{G_i}$, $\bar{h} \in \overline{G_{i+1}}$ ~~then~~ let g, h be preimages in G_i, G_{i+1}.

Then $\bar{g}\bar{h}\bar{g}^{-1} = f(ghg^{-1}) \in \overline{G_{i+1}}$ since $ghg^{-1} \in G_{i+1}$

And consider map $G_i \xrightarrow{f|_{G_i}} \overline{G_i} \to \overline{G_i}/\overline{G_{i+1}}$ (both maps surjective)

Kernel K contains G_{i+1}.

so by corresp thm (third or isom thm):

$$\overline{G_i}/\overline{G_{i+1}} \cong G_i/K \cong (G_i/G_{i+1})/(K/G_{i+1}) \cong \text{quotient of an abelian gp}$$

so $\overline{G_i}/\overline{G_{i+1}}$ are abelian, $\overline{G} = G/N$ is solvable.

<u>Prop:</u> Let G be a gp, $N \triangleleft G$. Suppose N, G/N are solvable. Then so is G.

<u>Pf:</u> Let $\{e\} = N_0 \triangleleft N_1 \triangleleft N_2 \cdots \triangleleft N_r = N$ be a normal series with abelian quotients in N

Let $\{e\} = \overline{G_0} \triangleleft \overline{G_1} \triangleleft \cdots \triangleleft \overline{G_s} = G/N$ be a " " " " " in G/N.

By corresp. thm have $G_i < G$ s.t. $G_i \supset N$, s.t. $G_i \triangleleft G_{i+1}$, and $G_{i+1}/G_i \cong \overline{G_{i+1}}/\overline{G_i}$ are abelian (note: $G_0 = N$)

then $\{e\} = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_r = G_0 \triangleleft G_1 \triangleleft G_2 \cdots \triangleleft G_s = G$

is the desired series.

G_i/H is abelian: $G_i/H \simeq \underbrace{(G_i/G_{i+1})}_{\text{abelian}} / (H/G_{i+1})$

so refining ~~seu~~ the series by inserting H between G_i, G_{i+1} retains the property of having abelian quotient.

Since G is finite, finitely many refinements result in a ~~seq~~ composition series, where every factor is simple, hence an abelian simple gp, ie. cyclic of prime order

<u>Say</u>: $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_r = \{e\}$ is a normal series quotients G_i/G_{i+1} abelian.

<u>Lemma</u>: $H < G$, $N \triangleleft G$ then H, G/N are solvable

<u>Pf</u>: set $H_i = H \cap G_i$. Then $H_0 = H \supset H_1 \supset H_2 \cdots \supset H_r = \{e\}$

Since $G_{i+1} \triangleleft G_i$, by 2nd isom thm: $H_{i+1} \triangleleft H_i$ and

$$H_i/H_{i+1} \hookrightarrow G_i/G_{i+1}.$$

~~Di~~ <u>direct pf</u>: let $q: G_i \to G_i/G_{i+1}$ be the quotient map

$Ker(q\restriction_{H_i}) = \{h \in H \cap G_i \mid q(h) = e\} = H \cap Ker(q) = H \cap G_{i+1} = H_{i+1}$

1st isom thm: $H_i/H_{i+1} = H_i/Ker(q\restriction_{H_i}) \simeq Im(q\restriction_{H_i}) < \underbrace{G_i/G_{i+1}}_{\text{abelian}}$

so H_i/H_{i+1} ~~is~~ are abelian

so H is solvable

4 i $[b,a] = [a,b]^{-1}$ so $[A,B] = [B,A]$

~~Conversely~~ Also, if $f \in \mathrm{Hom}(G,H)$ then $f([a,b]) = [f(a), f(b)]$

so $\gamma^i(f(G)) = f(\gamma^i(G))$

image

so if G is nilp so is its image ~~f~~ (i.e. every quotient of a nilpotent group is nilpotent)

Summary: If G is nilpotent, every subgp & quotient is nilpotent (of a ~~sm~~ smaller class)

Also G has ~~another~~ series of subgps $Z^i(G)$, $\gamma^i(G)$ where every subgp is normal in next, quotients abelian.

Def: G any gp. A normal series in G is a sequence of subgps

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_r = \{e\}$$

note: G_{i+1} is normal in G_i, not necc. in G.

Say this is a composition series if G_i/G_{i+1} are simple for all i.

(Every finite gp has one)

Thm (Schur-Zassenhaus): The "composition factors" G_i/G_{i+1}

Last time: nilpotence

G gp, defined $Z^0(G) = \{e\}$, $Z^{i+1}(G)/Z^i(G) = Z(G/Z^i(G))$

$\gamma^0(G) = G$, $\gamma^{i+1}(G) = [G, \gamma^i(G)]$.

~~saw~~ saw G nilpotent iff $Z^r(G) = G$ iff $\gamma^r(G) = \{e\}$ for some $r \geq 0$.

(Def way: $\{e\}$ is nilp. of deg 0, G is nilp of deg $s+1$ if $G/Z(G)$ is nilp of deg s)

One feature of these series: $\{e\} = Z^0(G) \subset Z^1(G) \subset \dots Z^r(G) = G$

$G = \gamma^0(G) \supset \gamma^1(G) \supset \dots \gamma^r(G) = \{e\}$

these subgps all normal in G, and quotients $Z^{i+1}(G)/Z^i(G)$, $\gamma^i(G)/\gamma^{i+1}(G)$

are commutative.

(recall: $[A,B] = \langle \{[a,b] = aba^{-1}b^{-1}\}_{\substack{a\in A\\ b\in B}} \rangle$)

Cor: Say $H < G$. Then $\gamma^i(H) < \gamma^i(G)$ by induction on i.

$\Rightarrow$ if G nilpotent so is H.

($\gamma^1(G) = [G,G] \subset G$ so $\gamma^2(G) = [G, \gamma^1(G)] \subseteq [G,G] = \gamma^1(G)$)

In a composition series are unique up to permutation.

Ex: The unique composition series of S_n $(n \geq 5)$
is $\{e\} \triangleleft A_n \triangleleft S_n$, factors are A_n, $C_2 \cong S_n/A_n$

Ex: G, H simple (non-isom) $G \times H$ has two composition series:

$\{e\} \triangleleft G \times \{e\} \triangleleft G \times H$

$\{e\} \triangleleft \{e\} \times H \triangleleft G \times H$

Def: Call G solvable if G has a normal series (Galois) with abelian quotients

Thm (Galois) let $f \in F[x]$, irred, char$(F)=0$. Then roots of f can be expressed using radicals iff $Gal(f)$ is solvable

Note: If G is finite: G solvable iff its composition factors are cyclic

Note: If G is nilpotent, G is solvable

Pf: Say $G = G_0 \triangleright G_1 \triangleright G_2 \cdots \triangleright G_r = \{e\}$ is a normal series with abelian quotients. Say G_i/G_{i+1} not simple:
eg. $\{e\} \neq H/G_{i+1} \underset{\neq}{\triangleleft} G_i/G_{i+1}$, By correspondence thm,
H/G_{i+1} corresponds to a subgp $G_{i+1} \subsetneq H \underset{\neq}{\triangleleft} G_i$.
then $G_{i+1} \triangleleft H$ (it's normal in G_i), H/G_{i+1} abelian (subgp of G_i/G_{i+1})