Math 322, lecture 22, 23/11/2017

Nilpotence

Def: Call G nilpotent of order 0 if $G = \{e\}$

Call G nilpotent of order $(k+1)$ if $G/Z(G)$ is nilpotent of order k.

(Call G nilpotent if it's nilp. of order k for some $k \geq 0$)

(In linear algebra, $T \in \mathrm{End}_F(V)$ is nilpotent if $T^k = 0$ for some k)

Ex: G is nilpotent of order 1 iff $G/Z(G) = \{e\}$ iff $G = Z(G)$ iff G is abelian.

G is nilpotent of order 2 iff $G/Z(G)$ is abelian

$\rightsquigarrow$

"Opposite": $Z(G) = \{e\}$

HW: the images of $x, y \in G$ in G/N $(N \triangleleft G)$ commute in G/N iff $[x,y] = xyx^{-1}y^{-1} \in N$.

(i.e. $Z(G/Z(G))$ need not be trivial) ↖ commutator of x,y.

So define $Z^0(G) = \{e\}$, $Z^1(G) = Z(G)$, $Z^2(G) =$ the preimage in G of $Z(G/Z(G))$

Generally, set $Z^{i+1}(G) =$ the preimage in G of $Z(G/Z^i(G))$.

$=$ the subgp of G containing $Z^i(G)$ s.t. $Z^{i+1}(G)/Z^i(G) \subset G/Z^i(G)$ is the centre

(Since $Z(G/Z^i(G))$ is normal in $G/Z^i(G)$, by correspondence thm, $Z^{i+1}(G)$ is normal in G)

Now $G = Z^k(G)$ iff G is nilpotent of order k

but $G \neq Z^{k-1}(G)$

Note: $Z^{i+1}(G)/Z^i(G)$ is abelian. (it's centre of $G/Z^i(G)$)

Have an increasing series of normal subgps:

$\{e\} = Z^0(G) \subset Z^1(G) \subset Z^2(G) \cdots$

Called: <u>increasing central series.</u>

(ascending)

<u>Example</u>: F field, $U_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \subset GL_n(F)$

$Z(U_n) = \left\{ \begin{pmatrix} 1 & & 0 & * \\ & \ddots & & 0 \\ 0 & & & 1 \end{pmatrix} \right\}$ $\leftarrow$ pos $1,n$

reason: let $\epsilon^{ij} = \begin{pmatrix} & \overset{j \downarrow}{1} & \end{pmatrix} \leftarrow i$

then $\epsilon^{ij}\epsilon^{kl} = \begin{cases} 0 & j \neq k \\ \epsilon^{il} & j = k \end{cases}$

$(1,n-1),(1,n)$ b $(2,n)$

$Z^2(U_n) = \left\{ \begin{pmatrix} 1 & & 0 & * & * \\ & \ddots & & & * \\ 0 & & & & 1 \end{pmatrix} \right\}$

<u>Ex</u>: this continues: $Z^k(U_n) = \left\{ u \;\middle|\; u_{ij} = \begin{cases} 0 & j < i \\ 1 & j = i \\ 0 & i+1 \leq j \leq n-k+i-1 \\ * & \text{otherwise} \end{cases} \right\}$

= last k diagonals free, others are zero

U_n is nilpotent of order $n-1$

$U_2 = \left\{\begin{pmatrix}1 & * \\ & 1\end{pmatrix}\right\} \cong (\mathbb{F}, +)$

$U_3 = \left\{\begin{pmatrix}1 & x & z \\ & 1 & y \\ & & 1\end{pmatrix}\right\}$ "Heisenberg group".

__Note:__ If $u \in U_n$ then $(u - \mathrm{Id})^n = 0$ (so u is nilpotent as a linear map)

One application of nilpotence: study matrix gps using nilpotent subgroups

__Example:__ finite p-groups are nilpotent:

If $\# G = p^n$, G nilpotent of degree $\leq n$

__Pf:__ By induction on n: ~~$Z(G)$~~ if $n \geq 1$, $Z(G) \neq \{e\}$, $G/Z(G)$ is a smaller p-group.

__Example:__ $Z(G \times H) = Z(G) \times Z(H)$ (show this!)

$\Rightarrow G \times H / Z(G \times H) \cong G/Z(G) \times H/Z(H)$

use to show: G, H nilp. $\Rightarrow$ $G \times H$ is nilpotent.

__Thm:__ (PS10, extra credit) Let G be a finite nilpotent group. Then G has a unique p-Sylow subgp for each $p \mid \# G$, and $G \cong \prod_p P_p$.

The descending central series

Def: A central series in G is a sequence of normal subgps: $\{e\} = G_0 < G_1 < G_2 \cdots < G_r = G$

s.t. ${}^{G_{i+1}}/_{G_i} \subseteq Z({}^{G}/_{G_i})$

Claim: let $\{G_i\}$ be a central series in G. Then $G_i \subset Z^i(G)$

Cor: G nilpotent iff it has a central series

(the ascending central series is a central series, conversely if $G_r = G$ then $Z^r(G) = G$)

Pf: Suppose $G_i \subset Z^i(G)$ (true for $i=0$)

Then $G_{i+1}/G_i \subset Z({}^{G}/_{G_i})$, ${}^{Z^{i+1}(G)}/_{Z^i(G)} = Z({}^{G}/_{Z^i(G)})$

For $z \in G_{i+1}$ to be in $Z^{i+1}(G)$ would mean: for every $g \in G$, z and g commute modulo $Z^i(G)$.

We know: z commutes with g modulo G_i.

So we know: $[z,g] \in G_i$, want: $[z,g] \in Z^i(G)$

but we assumed by induction that $G_i \subset Z^i(G)$ so $z \in Z^{i+1}(G)$ and $G^{i+1} \subset Z^{i+1}(G)$ ■

try to go from top: $G_r = G$ satisfies: $G_r/G_{r-1} = Z({}^{G}/_{G_{r-1}})$

but $G = G_r$, so G/G_{r-1} is abelian

So every x, y in G commute mod G_{r-1}, so $G_{r-1} \supseteq \{[x,y] : x,y \in G\}$

Def: The *derived* or *commutator* subgp of G is the subgp $G' = G^{(1)} = [G,G] = \langle \{[x,y] : x, y \in G\} \rangle$

note subgp generated by commutators.

G/N abelian iff $N \supseteq G'$

Also, $[G,G] \trianglelefteq G$.

Next, $G_{r-1}/G_{r-2} \subseteq Z(G/G_{r-2})$

so if $z \in G_{r-1}$, $g \in G$, $[z,g] \in G_{r-2}$

so $G_{r-2} \supseteq \left\{ [z,g] \;\middle|\; \begin{matrix} z \in G_{r-1} \\ g \in G \end{matrix} \right\} \supseteq \left\{ [z,g] \;\middle|\; \begin{matrix} z \in [G,G] = G^{(1)} \\ g \in G \end{matrix} \right\}$

again $G_{r-2} \supseteq \left\langle \left\{ [z,g] \;\middle|\; \begin{matrix} z \in G^{(1)} \\ g \in G \end{matrix} \right\} \right\rangle \overset{def}{=} [G^{(1)}, G] = \gamma^2(G)$

(in general, $[H,K] = \left\langle \left\{ [h,k] : \begin{matrix} h \in H \\ k \in K \end{matrix} \right\} \right\rangle$)

Continuing by induction, same argument shows:

$$G_{r-i} \supseteq \gamma^i(G) \quad \text{where} \quad \gamma^1(G) = [G,G]$$
$$\gamma^{i+1}(G) = [\gamma^i(G), G].$$

(note: if $\gamma^i(G) \leq \gamma^{i-1}(G)$ then $[\gamma^i(G), G] \subseteq [\gamma^{i-1}(G), G]$

so $G = \gamma^0(G) \supset \gamma^1(G) \supset \gamma^2(G) \supset \cdots$

also, $[gHg^{-1}, gKg^{-1}] = g[H,K]g^{-1}$ because $([ghg^{-1}, gkg^{-1}] = g[h,k]g^{-1})$

So if H, K normal, so is $[H,K]$. By induction, $\gamma^i(G)$ all normal.

Finally, consider $\gamma^{i}(G)/\gamma^{i+1}(G) \subset G/\gamma^{i+1}(G)$

if $z \in \gamma^i(G)$, $g \in G$ then $[z,g] \in \gamma^{i+1}(G)$

so $\gamma^{i}(G)/\gamma^{i+1}(G) \subset Z(G/\gamma^{i+1}(G))$

i.e. $\gamma^i(G)$ is a central series - fastest descending one

so G is nilpotent iff $\gamma^k(G) = \{e\}$ for some k, smallest such k is the degree of nilpotence

(in fact, G is nilpotent of order k iff forall $x_1, .., x_{k+1} \in G$:

$[[[.. [x_1, x_2], x_3], .., x_k], x_{k+1}] = e$),

Ex: let G be nilpotent. Then $X \subset G$ generates G iff image of X generates G/G'.

Ex: G nilp. Then $G_{tors} = \{g \in G \mid g \text{ has finite order}\}$ is a subgp.