Math 322, lecture 13, 24/10/17

Goal: Generalise last lecture.

Fix gp G acting on a set X.

Def: Say x,y in the same orbit if $\exists g \in G : gx = y$.

Lemma: This is an equivalence relation

Def: Classes called orbits (of G on X), write $G\backslash X$ for set of orbits, $G \cdot x$ or $O(x)$ for orbit of x.

Def: The stabilizer of $x \in X$ is $\mathrm{Stab}_G(x) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot x = x\}$

Lemma: $\mathrm{Stab}_G(x)$ is a subgp of G.

Pf: $e \cdot x = x$ (axiom for actions) if $g \cdot x = x$ then

$g^{-1} \cdot x = g^{-1}(g \cdot x) = (g^{-1}g) \cdot x = e \cdot x = x$,

and if $g \cdot x = x$, $h \cdot x = x$ then $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$.

Prop: (Orbit-stabilizer thm) (1) the Map $g\mathrm{Stab}_G(x) \mapsto g \cdot x$ is a bijection $G/\mathrm{Stab}_G(x) \to O(x)$.

(2) $\mathrm{Stab}_G(gx) = g\,\mathrm{Stab}_G(x)\,g^{-1}$, and $\{\mathrm{Stab}_G(y)\}_{y \in O(x)}$ is a conjugacy class of subgroups

Pf: (1) The map is well-defined since if $g' = g \cdot s$, $\underset{s \in \mathrm{Stab}_G(x)}{s \cdot x = x}$

then $g' \cdot x = (gs)x = g(sx) = g \cdot x$.

is surjective since $g\cdot x$ is image of $g\,\mathrm{Stab}_G(x)$

It is injective since if $g\cdot x = g'x$ then $(g^{-1}g')\cdot x = x$
so $g^{-1}g' \in \mathrm{Stab}_G(x)$ and $g\,\mathrm{Stab}_G(x) = g'\,\mathrm{Stab}_G(x)$

(2) $h\cdot(g\cdot x) = g\cdot x$ iff $g^{-1}\cdot(h\cdot(g\cdot x)) = g^{-1}\cdot(g\cdot x)$

$\Updownarrow$ iff $(g^{-1}hg)\cdot x = x$ iff $g^{-1}hg \in \mathrm{Stab}_G(x)$

$h \in \mathrm{Stab}_G(gx)$ iff $h \in g\,\mathrm{Stab}_G(x)\,g^{-1}$.

Reading left-to-write: if $y = g\cdot x$ then $\mathrm{Stab}_G(y)$ is conjugate to $\mathrm{Stab}_G(x)$ by g.

Reading right-to-left: if H is conjugate to $\mathrm{Stab}_G(x)$ by g then $H = \mathrm{Stab}_G(g\cdot x)$

<u>Cor</u>: (General class equation):

$$\# X = \sum_{\mathcal{O}(x)\in G\backslash X} \#\mathcal{O}(x) = \sum_{\mathcal{O}(x)\in G\backslash X} [G : \mathrm{Stab}_G(x)]$$

<u>Application:</u> <u>Def</u>: $\mathrm{Fix}(G) = \{x \in X \mid \mathrm{Stab}_G(x) = G\}$
$= \{x \in X \mid \mathcal{O}(x) = \{x\}\}$

<u>Cor</u>: Suppose $\# G = p^k$, X is finite.

Then $\# X \equiv \#\mathrm{Fix}(G) \ (p)$

<u>Pf</u>: $\# X = \sum_{\mathcal{O}(x)\in G\backslash X} \#\mathcal{O}(x) = \sum_{x\in\mathrm{Fix}(G)} 1 + \sum_{\mathcal{O}(x)\in G\backslash X,\ \mathrm{Stab}_G(x)\neq G} [G : \mathrm{Stab}_G(x)]$

$$= \# \mathrm{Fix}(G) + \sum_{\substack{\mathcal{O}(x) \in G \backslash X \\ \mathrm{Stab}_G(x) \neq G}} [G : \mathrm{Stab}_G(x)]$$

Now by Lagrange's thm, $[G : \mathrm{Stab}_G(x)] \mid \# G$, so if not 1 this index is a power of p and the claim follows

Application (Zagier): An involution on X is a permutation of order 2.

$\Leftrightarrow$ action of C_2 on X

Cor: An involution on a set of odd size has a fixed point.

Cor: If X admits an involution, with odd # of fixed points then $\# X$ is odd

Let p be a prime, $p \equiv 1 \ (4)$

Let $X = \left\{ (x,y,z) \in \mathbb{Z} \;\middle|\; \begin{array}{l} x^2 + 4yz = p \\ 0 < x,y,z < p \end{array} \right\}$

X has an obvious involution $\sigma(x,y,z) = (x,z,y)$

Zagier: X has a non-obvious involution τ, $\#\mathrm{Fix}(\tau) = 1$

So $\# X$ is odd, so σ has a fixed point (x,y,y)

i.e. can write $p = x^2 + 4y^2$

(Thm due to Fermat)

let G be a p-gp, say G acts on $\mathbb{F}_p^k$ by linear maps.

($\mathbb{Z}/p\mathbb{Z}$ as a field)

$\# \mathbb{F}_p^k = p^k$ so $p \mid \# \mathrm{Fix}(G)$.

But $\underline{0} \in \mathrm{Fix}(G)$ so $\# \mathrm{Fix}(G) \geq p$ and G fixes a non-zero vector.

Examples; Actions, orbits, stabilizers

(1) G acting on G/H.

G is a gp, $H < G$, $X = G/H$, set $g \cdot C = gC$ if $C \in G/H$ $\quad gC = \{gx \mid x \in C\}$

ie $g \cdot xH = (gx) \cdot H$.

check; $e \cdot xH = exH = xH$, $(gh) \cdot C = ghC = g(hC) = g \cdot (h \cdot C)$.

- Orbits: if $xH, yH \in G/H$, then $(yx^{-1}) \cdot xH = yH$,

  so only one orbit.

  Say action is transitive.

- Stabilizers; $\mathrm{Stab}_G(H) = \{g \in G \mid gH = H\} = H$.

(for any $H < G$ have a transitive action of G with point stabilizer H)

Prop; Let G act on X then the map $gH \mapsto g \cdot x$

$(H = \mathrm{Stab}_G(x))$, is a map of G-sets $G/H \to X$: $f(gH) = g \cdot x$

$$f(g\cdot C) = g\cdot f(C)$$
for any $g \in G$, $C \in G/H$.

---

(2) $GL_n(\mathbb{R})$ acting on $\mathbb{R}^n$

- For $g \in GL_n(\mathbb{R})$, $\underline{v} \in \mathbb{R}^n$ write $g\cdot\underline{v}$ for matrix vector product.

. This is an action.

- <u>Orbits</u>: $\{\underline{0}\}$ is clearly an orbit.

$\mathbb{R}^n \setminus \{\underline{0}\}$ is an orbit:

Let $\underline{u}$, $\underline{v}$ be non zero. Choose bases $\{\underline{u}_i\}_{i=1}^n$, $\{\underline{v}_i\}_{i=1}^n$ of $\mathbb{R}^n$ with $\underline{u}_1 = \underline{u}$, $\underline{v}_1 = \underline{v}$

Then the unique linear map T s.t. $T\underline{u}_i = \underline{v}_i$ is invertible, and has $T\underline{u} = \underline{v}$

- <u>Stabilizers</u>; $\mathrm{Stab}(\underline{0}) = GL_n(\mathbb{R})$

$$\mathrm{Stab}_{GL_n}(\underline{e}_n) = \left\{ g \;\middle|\; g\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix} \right\} =$$

$$= \left\{ g = \begin{pmatrix} h & \underline{0} \\ \underline{u} & 1 \end{pmatrix} \;\middle|\; \begin{matrix} h \in GL_{n-1}(\mathbb{R}) \\ \underline{u} \in \mathbb{R}^{n-1} \end{matrix} \right\}$$

---

(3) $GL_n(\mathbb{R})$ acting on $\mathbb{R}^n \times \mathbb{R}^n$, diagonally.

$$g\cdot(\underline{u},\underline{v}) = (g\underline{u}, g\underline{v})$$

Orbits: $\{(\underline{0},\underline{0})\}$, $\{(\underline{0},\underline{v}) \mid \underline{v} \neq \underline{0}\}$, $\{(\underline{u},\underline{0}) \mid \underline{u} \neq \underline{0}\}$

$\{(\underline{u}, c\,\underline{u}) \mid \underline{u} \neq \underline{0}\}$ for $c \neq 0$

$\{(\underline{u}, \underline{v}) \mid \{\underline{u}, \underline{v}\}$ indep set $\}$