Math 322, lecture 5, 21/9/2017

Last time: $S_X = \{\sigma : X \to X \mid \sigma \text{ bijective}\}$

$S_n = S_{\{1,2,\dots,n\}}$ | $\sigma, \tau \in S_X \Rightarrow \sigma\tau, \sigma^{-1} \in S_X$.

r-cycle $(i_1\, i_2\, i_3 \dots i_r)$ is the map $\kappa(i) = \begin{cases} i_{j+1} & i = i_j,\ j<r \\ i_1 & i = i_r \\ i & i \neq i_j \text{ all } j \end{cases}$

Thm: Every $\sigma \in S_n$ is of the form $\sigma = \prod_j \kappa_j$, κ_j disjoint cycles, unique up to reordering

Pf: For $i,j \in \{1,\dots,n\}$ set $i \sim j$ if $i = \sigma^k(j)$ for some $k \in \mathbb{Z}$

check: ① $\sim$ is an equivalence relation

② each class is σ-invariant: if $i \in$ class, $\sigma(i)$ also there

③ Define cycles by restricting σ to classes

Example: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 2 & 1 & 5 & 7 & 4 \end{pmatrix}$ $\quad 1 \sim 6 \sim 7 \sim 4 \sim 1$

$2 \sim 3$

5

Today:

(1) sign of a permutation

(2) linear groups

Odd and even permutations

Lemma: Every permutation is a product of 2-cycles ("transpositions")

[the transpositions generate S_n]

Pf: let $\sigma \in S_n$ be a counterexample of minimal support.

$\sigma \neq id$ ($id = ()$) so $supp(\sigma) \neq \phi$. Say $i \in supp(\sigma)$

Consider permutation $\tau = (i\ \sigma(i)) \circ \sigma$ (apply σ, swap $i, \sigma(i)$)

$i \neq \sigma(i)$ since $i \in supp(\sigma)$)

if $j \notin supp(\sigma)$ then $j \neq i$, $j \neq \sigma(i)$ so $\tau(j) = (i\ \sigma(i))(\sigma(j))$
$= (i\ \sigma(i))(j) = j$

also, $\tau(i) = (i\ \sigma(i))(\sigma(i)) = i$ so $i \notin supp(\tau)$

so $supp(\tau) \subset supp(\sigma) \setminus \{i\}$

so τ is a pdt of transpositions: $\tau = \prod_\ell \beta_\ell$.

Then $(i\ \sigma(i)) \cdot \sigma = \prod_\ell \beta_\ell$

Then $\underbrace{(i\ \sigma(i))^2}_{} \sigma = (i\ \sigma(i)) \cdot \prod_\ell \beta_\ell$

(let $\beta = (i\ \sigma(i)) = 1$ then let $\tau = \beta\sigma$, for $j \notin supp(\sigma)$, $\tau(j) = \ldots$)

Pf: Know $\sigma \in S_n$ is pdt of cycles enough to show each cycle is a pdt of transpositions

By induction: $(i_1, \ldots, i_r) = (i_1\ i_2)(i_2\ i_3)(i_3\ i_4) \cdots (i_{r-1}\ i_r)$

Def: The alternating group A_n is the set of elements that are the product of an even number of transpositions (those permutations are said to be "even")

Remark: If $\sigma, \tau \in A_n$, so do $\sigma\tau$, σ^{-1}

(concatenate or reverse even products keeps them even)

Lemma: let $\cancel{1} \le k \le n$ then in S_8

$$(a_1\ a_k)(a_1, \ldots, a_n) = (a_1 \ldots a_{k-1})(a_k \ldots a_n)$$

(if $k=1$ or $k=2$, $(a_1\ a_1)$ means id)

$$(a_1\ a_k)(a_1, \ldots, a_{k-1})(a_k \ldots a_n) = (a_1, \ldots, a_n)$$

Pf: ~~1~~st by direct evaluation

2nd follows by multiplying by $(a_1\ a_k)$ on left, using $(a_1\ a_k)^2 = id$

Def: For $\sigma \in S_n$, let $\sigma = \prod_{j=1}^{t} K_j$ be the cycle decomposition of σ (add a 1-cycle for each fixed point)

Set $sgn(\sigma) = (-1)^{n-t}$ call it the sign of σ.

Key Lemma: let τ be a transposition. Then

$$sgn(\tau\sigma) = -sgn(\sigma)$$

Pf: Say $\tau = (a_1\ a_k)$ either, a_1 & a_k both in same cycle for σ or they are in different cycles (if only one, assume its K_1, if two assume its K_1, K_2)

~~Lemma~~ previous lemma: in either case, # of cycles in $\tau\sigma$ changes by one

(in case 1, $\tau\sigma = (a_1 \ldots a_{k-1})(a_k \ldots a_n) \cdot (K_2 K_3 \cdots K_t$

2: $\tau\sigma = (a_1, \ldots a_n) \cdot K_3 K_4 \cdots K_t$

Thm: For all $\sigma, \tau \in S_n$ have $\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)$

[interpretation: map $\mathrm{sgn}: S_n \to \{\pm 1\}$ respects multiplication]

Pf: key lemma showed $\mathrm{sgn}(\tau\sigma) = \mathrm{sgn}(\tau)\,\mathrm{sgn}(\sigma)$ if τ is a transposition

Set $H = \{\tau \mid \forall \sigma: \mathrm{sgn}(\tau\sigma) = \mathrm{sgn}(\tau)\,\mathrm{sgn}(\sigma)\}$.

We know H contains all transpositions. Also, if $\tau_1, \tau_2 \in H$ then $\tau_1\tau_2 \in H$ because for all σ:

$$\mathrm{sgn}((\tau_1\tau_2)\sigma) \underset{\substack{\uparrow \\ \text{associativity} \\ \text{in } S_n}}{=} \mathrm{sgn}(\tau_1(\tau_2\sigma)) \underset{\substack{\uparrow \\ \tau_1 \in H}}{=} \mathrm{sgn}(\tau_1)\,\mathrm{sgn}(\tau_2\sigma) =$$

$$\overset{\tau_2 \in H}{=} \mathrm{sgn}(\tau_1)\,\mathrm{sgn}(\tau_2)\,\mathrm{sgn}(\sigma) \overset{\tau_1 \in H}{=} \mathrm{sgn}(\tau_1\tau_2)\,\mathrm{sgn}(\sigma).$$

But every element of S_n is a prod of transpositions, so $H = S_n$.

Cor: Suppose $\sigma = \prod_{j=1}^{s} \tau_j$, τ_j are transpositions

Then $\mathrm{sgn}(\sigma) = \mathrm{sgn}\left(\prod_{j=1}^{s} \tau_j\right) = \prod_{j=1}^{s} \mathrm{sgn}(\tau_j) = (-1)^s$

so parity of s depends only on σ.

Cor: $(n \geq 2)$ $\# A_n = \frac{1}{2} \# S_n$ Pf: Let τ be a transposition. Then mult by τ swaps A_n, $S_n \setminus A_n$

Ex: Saw that r-cycle has sgn $(-1)^{r-1}$!

Show A_n is generated by 3-cycles $(n \geq 3)$