

Lior Silberman's Math 322: Problem Set 10 (due 28/11/2017)

- P1. Find a group G and three pairwise disjoint subgroups A, B, C such that the multiplication map $A \times B \times C \rightarrow G$ is not injective.

DEFINITION. Let G be a group. Call $g \in G$ a *torsion element* if g has finite order ($g^k = e$ for some $k \neq 0$), and write G_{tors} for the set of torsion elements. Say that g is *p-power torsion* if its order is a power of p . For an abelian group write $A[n]$ for the set of elements of order dividing n and $A[p^\infty]$ for the set p -power torsion elements.

- P2. (Torsion) Let G, H be groups, A an abelian group.
- If G is finite then $G = G_{\text{tors}}$. Give an example of an infinite group consisting entirely of torsion elements.
 - Show that $f(G_{\text{tors}}) \subset H_{\text{tors}}$ for any $f \in \text{Hom}(G, H)$.
 - $A_{\text{tors}} = \bigcup_{n \geq 1} A[n]$, $A[p^\infty] = \bigcup_{r=0}^{\infty} A[p^r]$.
 - Let $X \in \text{GL}_n(\mathbb{R})$ be a torsion element. Show that the eigenvalues of X are (possibly complex) roots of unity.
 - Find $X, Y \in \text{GL}_n(\mathbb{R})_{\text{tors}}$ such that XY has infinite order.

Abelian groups

- (First do problem P2) Fix an abelian group A .
 - Show that A_{tors} and $A[p^\infty]$ are subgroups of A .
 - Show that $A[p^\infty]$ is the p -Sylow subgroup of A .
— It follows that, if A is finite, $A = \prod_p A[p^\infty]$ as an internal direct product.
 - Show that A/A_{tors} is *torsion-free*: $(A/A_{\text{tors}})_{\text{tors}} = \{e\}$.
- Find the isomorphism classes of the Sylow subgroups of $C_{360} \times C_{300} \times C_{200} \times C_{150}$.

Nilpotent groups and torsion

- Let G be *two-step nilpotent*, in that $G/Z(G)$ is abelian.
PRAC Verify that the Heisenberg group (PS7 problem P2) is two-step nilpotent.
 - For $x, y \in G$ let $[x, y] = xyx^{-1}y^{-1}$ be their commutator. Show that $[x, y] \in Z(G)$ for all G (hint: this is purely formal).
 - Let $x, y \in G$ and $z, z' \in Z(G)$. Show that $[x, y] = [xz, yz']$ and conclude that the commutator induces a map $G/Z \times G/Z \rightarrow Z$.
 - Show that this map is *anti-symmetric*: $[\bar{y}, \bar{x}] = [\bar{x}, \bar{y}]^{-1}$ and *biadditive*: $[\bar{x}\bar{x}', \bar{y}] = [\bar{x}, \bar{y}][\bar{x}', \bar{y}]$, $[\bar{x}, \bar{y}\bar{y}'] = [\bar{x}, \bar{y}][\bar{x}, \bar{y}']$.
 RMK In fact, a two-step nilpotent group is more-or-less determined by the abelian groups $A = G/Z(G)$, $Z = Z(G)$ and the anti-symmetric biadditive form $[\cdot, \cdot] : A \times A \rightarrow Z$.
- (Torsion in nilpotent groups) Continue with the hypotheses of problem 3.
 - Let $x, y \in G$ and suppose that $x \in G_{\text{tors}}$. Show that $[x, y] \in Z(G)_{\text{tors}}$.
 - (**b) (The hard part). Show that G_{tors} is a subgroup of G .

Extra credit

In general, a group is 0-step nilpotent if it is trivial, $(d + 1)$ -step nilpotent if $G/Z(G)$ is d -step nilpotent, and *nilpotent* if it is d -step nilpotent for some d . A variant on the argument of problem 4 shows that the set of torsion elements of any nilpotent group is a subgroup.

5**. Let G be a finite nilpotent group.

- (a) Show that the Sylow subgroups of G are normal.
- (b) Show that a finite group is nilpotent iff it is the direct product of its Sylow subgroups.