Lior Silberman's Math 322: Problem Set 9 (due 21/11/2017)

Practice Problem

P1. In class we classified the groups of order 12, finding the isomorphism types A_{12}, C_{12}, $C_4 \ltimes C_3$, $C_2 \times C_6$, $C_2 \ltimes S_6$. The dihedral group D_{12} is a group of order 12 – where does it fall in this classification?

P2. (Numerology) Let G be a group of order p^2q where p,q are prime.
(a) Show that, unless $q \equiv 1\,(p)$, G has a unique p-Sylow subgroup and isn't simple.
(b) Show that, unless $p^2 \equiv 1\,(q)$, G has a unique q-Sylow subgroup and isn't simple.
(c) Show that if $q \equiv 1\,(p)$ and $p^2 \equiv 1\,(q)$ then $p = 2, q = 3$ and G isn't simple.

Sylow's Theorems

Write P_p for a p-Sylow subgroup of G.

1. Let G be a simple group of order $36 = 2^2 3^2$.
RMK The idea of P2 shows that a group of order p^2q^2 isn't simple unless $p^2q^2 = 36$.
(a) Show that G acts non-trivially on a set of size 4.
(b) Use the kernel of the action to show G isn't simple after all.

2. Let G be a group of order $255 = 3 \cdot 5 \cdot 17$.
(a) Show that $n_{17}(G) = 1$.
(*b) Show that P_{17} is central in G.
(*c) Show that $n_5(G) = 1$.
(d) Show that P_5 is also central in G.
(e) Show that $G \simeq C_3 \times C_5 \times C_{17} \simeq C_{255}$.

3. Let G be a group of order 140
(a) Show that $G \simeq H \ltimes C_{35}$ where H is a group of order 4.
(*b) Classify actions of C_4 on C_{35} and determine the isomorphism classes of groups of order 140 with $P_2 \simeq C_4$.
(**c) Classify actions of V on C_{35} and determine the isomorphism classes of groups of order 140 with $P_2 \simeq V$.

4. Let G be a finite group, $P < G$ a Sylow subgroup. Show that $N_G(N_G(P)) = N_G(P)$ (hint: let $g \in N_G(N_G(P))$ and consider the subgroup gPg^{-1}).

5. Let G be a finite group of order n, and for each $p|n$ let P_p be a p-Sylow subgroup of G.
(a) Show that $G = \left\langle \bigcup_{p|n} P_p \right\rangle$.
(b) Suppose that G_p has a unique p-Sylow subgroup for each p. Show that $G = \prod_p P_p$ (internal direct product).

(hint for 2b: conjugation gives a homomorphism $G \to \mathrm{Aut}(P_{17})$).
(hint for 2c: let G act by conjugation on $\mathrm{Syl}_5(G)$ and use part b).
(hint for 3b: use problem 6 of PS8 and the fact that $\mathrm{Aut}(C_p) \simeq C_{p-1}$).