Lior Silberman's Math 322: Problem Set 5 (due 12/10/2017)

Practice problems

P1. Let $N < G$ satisfy for all $g \in G$ that $gNg^{-1} \subset N$. Show that for all $g \in G$, $gNg^{-1} = N$.

P2. Let $N < G$ satisfy for all $g_1, g_2 \in G$ that if $g_1 \equiv_L g_1' \,(N)$ and $g_2 \equiv_L g_2' \,(N)$ then $g_1 g_2 \equiv_L g_1' g_2' \,(N)$.
   (a) Show that for any $g \in G$, $n \in N$ we have $gng^{-1} \equiv_L e \,(N)$, and conclude that $gNg^{-1} = N$.
   (b) Give $G/\equiv_L (N)$ a group structure, and construct a homomorphism $q\colon G \to G/N$ such that $N = \mathrm{Ker}(q)$. Conclude that N is normal.

Cosets, normal subgroups and quotients

1. (Normalizers and centralizers) Let G be a group, $X \subset G$ a subset. The *centralizer* of X (in G) is $Z_G(X) = \{g \in G \mid \forall x \in X : gx = xg\}$ (in particular $Z(G) = Z_G(G)$ is called the *centre* of G). The *normalizer* of X (in G) is $N_G(X) = \left\{g \in G \mid gXg^{-1} = X\right\}$. Fix $H < G$.
   (a) Show that $N_G(X) < G$.
   PRAC Show that $Z_G(X) < N_G(X)$.
   (b) Show $H < N_G(H)$.
   PRAC Let $H < K < G$. Show that $H \lhd K$ iff $K \subset N_G(H)$. In particular, $H \lhd G$ iff $N_G(H) = G$.
   (c) Show that $Z(G)$ is a normal, abelian subgroup of G.
   PRAC Show that $H \cap Z_G(H) = Z(H)$, in particular that $H \subset Z_G(H)$ iff H is abelian.

2. (Semidirect products) Let $H, K < G$ and consider the function $f\colon H \times K \to G$ given by $f(h,k) = hk$. The image of this function is usually denoted HK.
   (a) Show that f is injective iff $H \cap K = \{e\}$.
   SUPP For $x \in HK$ give a bijection $f^{-1}(x) \leftrightarrow H \cap K$, hence a bijection $H \times K \leftrightarrow HK \times H \cap K$.
   PRAC Show $H < N_G(K) \iff \forall h \in H : hKh^{-1} = K$. In this case we say "$H$ normalizes K".
   (b) Suppose H normalizes K. Show that HK is a subgroup of G and that $\langle H \cup K \rangle = HK$. Show that $K \lhd HK$ (hint: you need to show that $HK < N_G(K)$ and already know that H, K separately are contained there).
   DEF If $H < N_G(K)$ and $H \cap K = \{e\}$ we call HK the (*internal*) *semidirect product* of H and K. We write $HK = H \ltimes K$ (combining the symbols for product and normal subgroup).
   (c) Let HK be the semidirect product of H, K and let $q\colon HK \to (HK)/K$ be the quotient map. Directly show that the restriction $q \restriction_H \colon H \to (HK)/K$ is an isomorphism. (Hint: what is the kernel? what is the image?)
   PRAC Let $g, h \in G$. Show that $gh = hg$ iff the *commutator* $[g,h] = ghg^{-1}h^{-1}$ has $[g,h] = e$.
   — For parts (d),(e) suppose that H, K normalize each other and that $H \cap K = \{e\}$
   (d) Show that H, K *commute*: $hk = kh$ whenever $h \in H$, $k \in K$.
   (e) Show that the map f is an isomorphism onto its image (it's a bijection by part (a); you need to show it is a group homomorphism).
   DEF In that case we say HK is the (*internal*) *direct product* of H and K.

3. Let $K < H < G$ be a chain of subgroups. Let $R \subset G$ be a system of representatives for G/H and let $S \subset H$ be a system of representatives for H/K.
   (a) Show that the map $R \times S \to RS$ given by $(r,s) \mapsto rs$ is a bijection.
   (b) Show that $RS = \{rs \mid r \in R, s \in S\}$ is a system of representatives for G/K, and conclude that $[G:K] = [G:H][H:K]$.

   RMK See the practice problems file for a numerical proof in the finite case.

4. In a previous problem set we defined the subgroup $P_n = \{\sigma \in S_n \mid \sigma(n) = n\}$ of S_n. We now give an explicit description of S_n/P_n and use that to inductively determine the order of S_n.
   (a) Show that for $\tau, \tau' \in S_n$ we have $\tau P_n = \tau' P_n$ iff $\tau(n) = \tau'(n)$, and conclude that $[S_n : P_n] = n$.
   (b) Show that $P_n \simeq S_{n-1}$.
   (c) Combine (a),(b) into a proof by induction that $|S_n| = n!$.

Extra credit

5. Let G be a group
   (a) Suppose that $x^2 = e$ for all $x \in G$. Show that G is abelian.
   (b**) Suppose that G has n elements, at least $\frac{3}{4}n$ of which have order 2. Then G is abelian.

6**. Let G be group of order n. Show that there is $X \subset G$ of size at most $1 + \log_2 n$ such that $G = \langle X \rangle$.

Supplementary Problems: Quotients and the abelianization

A. (The universal property of G/N) Let $N \lhd G$. An "abstract quotient" of a group G is a group $\bar{G}$, together with a homomorphism $\bar{q}\colon G \to \bar{G}$ such that the property for any $f\colon G \to H$ with kernel containing N there is a unique $\bar{f}\colon \bar{G} \to H$ with $f = \bar{f} \circ \bar{q}$ (in class we saw that the quotient group G/N has this property). Suppose that $\left(\bar{G}', \bar{q}'\right)$ is another abstract quotient. Show that there is a uinque isomorphism $\varphi\colon \bar{G} \to \bar{G}'$ such that $\bar{q}' = \varphi \circ \bar{q}$.

B. (The derived subgroup and abelian quotients) Fix a group G and recall that notation $[g,h] = ghg^{-1}h^{-1}$.
   (a) Let $f \in \mathrm{Hom}(G,H)$ be a homomorphism. Show that $f([g,h]) = [f(g), f(h)]$ for all $g, h \in G$.
   (b) Deduce from (a) that the set of commutators is invariant under conjugation.

   DEF For $H, K < G$ set $[H,K] = \langle \{[h,k] \mid h,k \in G\} \rangle$ – note that this is the *subgroup* generated by those commutators, not just the set of commutators. In particular, we write $G' = [G,G]$ for the *derived subgroup* (or *commutator subgroup*) of G, the subgroup generated by all the commutators.

   (c) Show that G' is normal in G.
   (d) Show that $G^{\mathrm{ab}} \stackrel{\mathrm{def}}{=} G/G'$ is abelian (hint: apply (a) to the quotient map).

   DEF we call G^{ab} the *abelianization* of G.

   (e) Let $N \lhd G$. Show that G/N is abelian iff $G' \subset N$.
   (f) Let A be an abelian group and let $q\colon G \to G^{\mathrm{ab}}$ be the quotient map. Show that the map $\Phi\colon \mathrm{Hom}(G^{\mathrm{ab}}, A) \to \mathrm{Hom}(G, A)$ given by $\Phi(f) = f \circ q$ is a bijection.

C. Compute the derived subgroup and the abelianization of the groups: $C_n, D_{2n}, S_n, \mathrm{GL}_n(\mathbb{R})$.