Problem A runner jogs around circular track of radius 100m at the speed of $7\frac{\mathrm{km}}{\mathrm{h}}$. A friend is standing 200m from the center of the circle watching him. How fast is the distance between them changing when that distance is 200m?

Solution 1 Choose the co-ordinate system so that the origin is at the center of the circle and the friend is at $(200\mathrm{m}, 0)$. Let R be the radius of the circle, L the distance to the friend, V the velocity. Suppose that the runner is at position $(x, y) = (x(t), y(t))$ a time t. The data is then:

$$\begin{aligned} x^2 + y^2 &= R^2 && \text{running on the circle} && (1) \\ \dot{x}^2 + \dot{y}^2 &= V^2 && \text{fixed velocity} && (2) \end{aligned}$$

The distance $D = D(t)$ to the friend satisfies

$$\begin{aligned} D^2 &= (x - L)^2 + (y - 0)^2 \\ &= x^2 - 2xL + L^2 + y^2 \\ &= R^2 + L^2 - 2xL\,. && (3) \end{aligned}$$

Differentiating (note that R, L are constants) we find

$$2D \cdot \dot{D} = -2L\dot{x}$$

and hence

$$\dot{D} = -\frac{L}{D}\dot{x}\,. \qquad (4)$$

To find $\dot{x}$ we differentiate the constraint (1) to get $2x\dot{x} + 2y\dot{y} = 0$ and hence $\dot{y} = -\frac{x}{y}\dot{x}$. Plugging into the velocity constraint (2) we get

$$\dot{x}^2 + \left(-\frac{x}{y} \cdot \dot{x}\right)^2 = V^2$$

so

$$\left(1 + \frac{x^2}{y^2}\right)\dot{x}^2 = V^2$$

and

$$\dot{x}^2 = \frac{V^2 y^2}{x^2 + y^2} = \frac{V^2}{R^2}y^2\,.$$

Taking the square root we find

$$\dot{x} = \pm\frac{V}{R}y \qquad (5)$$

(don't know sign since we don't know whether the runner is running clockwise or anti-clockwise). Plugging (5) into (4) we get

$$\dot{D} = \pm\frac{Ly}{DR}V\,.$$

We'd like to write this as a function of D, which is given. From (3) we get that when the distance is D, the runner is at (x, y) with $x = \frac{R^2+L^2-D^2}{2L}$ and $y = \pm\sqrt{R^2 - x^2} = \pm R\sqrt{1 - \left(\frac{x}{R}\right)^2}$.

Measuring distances in metres, we're given $R = 100$, $L = D = 200$ so $x = 25$ and $\frac{y}{R} = \pm\frac{\sqrt{15}}{4}$ so $\dot{D} = \frac{\sqrt{15}}{7}V$. Inputting $V = 7$ we get

$$\dot{D} = \pm\frac{7\sqrt{15}}{4}\frac{\text{Km}}{\text{h}}\,.$$

Solution 2 With the same coordinate system suppose that the line between the runner and the origin makes angle α with the positive x-axis. The runner is then at $(R\cos\alpha, R\sin\alpha)$ and hence at distance

$$\begin{aligned} D^2 &= (R\cos\alpha - L)^2 + (R\sin\alpha)^2 \\ &= R^2\cos^2\alpha - 2LR\cos\alpha + L^2 + R^2\sin^2\alpha \\ &= R^2 + L^2 - 2LR\cos\alpha\,. \end{aligned}$$

Differentiating we find

$$2D\dot{D} = 2LR\,(\sin\alpha)\,\dot{\alpha}\,.$$

Since $V = R\dot{\alpha}$ (arclength is radius times angle), this means

$$\dot{D} = \frac{LV}{D}\sin\alpha\,,$$

so it remains to find $\sin\alpha$ at the given time. For this we use $\cos\alpha = \frac{R^2+L^2-D^2}{2LR} = \frac{1}{4}$ to get $\sin\alpha = \pm\sqrt{1-\cos^2\alpha} = \pm\frac{\sqrt{15}}{4}$ and again

$$\dot{D} = \pm\frac{7\sqrt{15}}{4}\frac{\text{Km}}{\text{h}}\,.$$