Math 100 §105, Fall Term 2010
Midterm Exam

November 8^{th},2010

Student number:

LAST name:

First name:

Instructions

- Do not turn this page over until instructed. You will have 45 minutes for the exam.
- You may not use books, notes or electronic devices of any kind.
- Solutions should be written clearly, in complete English sentences, showing all your work.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.

Signature:

1	/18
2	/6
3	/6
4	/10
Total	/40

1 Short-form answers

Show your work and clearly delineate your final answer. Not all problems are of equal difficulty.

[3] a. Differentiate the function $y = \arctan(x)$; write your answer as a function of x alone (you may use the formula $\frac{d\tan u}{du} = 1 + \tan^2 u$).

If $y = \arctan(x)$ then $\tan(y) = x$. Differentiating with respect to x and using the chain rule we find $1 = \frac{d(\tan y)}{dx} = (1 + \tan^2 y)\frac{dy}{dx} = (1 + x^2)\frac{dy}{dx}$. Solving for $\frac{dy}{dx}$ we find

$$\frac{dy}{dx} = \frac{1}{1 + x^2} .$$

[3] b. Find the tangent line to $y = x^{\sin x}$ at the point $(\pi, 1)$.

Taking logarithms we have $\ln y = \sin x \ln x$, and differentiating we find

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{\sin x}{x} .$$

When $x = \pi$ we have $y = 1$ and $\sin \pi = 0$ so

$$y'(\pi) = \cos \pi \ln \pi = -\ln \pi .$$

The tangent line is therefore

$$y = 1 - (x - \pi) \ln \pi .$$

[3] c. We have $z(t) = e^{x(t) \cdot y(t)}$ where x, y both depend on t. If at $t = 1$ we have $x(1) = 0$, $x'(1) = 1$, $y(1) = 2$, $y'(1) = 3$ find $\frac{dz}{dt}$ at $t = 1$.

Differentiating using the chain rule we have:

$$\begin{aligned} \frac{dz}{dt} &= e^{x(t)y(t)} \frac{d}{dt}(xy) \\ &= e^{xy}(x'y + y'x) . \end{aligned}$$

At $t = 1$ we have $x(1) = 0$ so

$$z'(1) = e^0 x'(1) y(1) = 1 \cdot 1 \cdot 2 = 2 .$$

[3] d. The population of Canada was roughly 18 million in the year 1960, 30 million in the year 2000. Assuming the population grows exponentially, estimate the population of Canada in 2040.

In 40 years the population grew by a factor of about $\frac{30}{18} = \frac{5}{3}$. We thus expect the population in 2040 to be about

$$\frac{5}{3} \cdot 30\text{million} = 50\text{million}\,.$$

[3] e. Let $f(x) = e^x + e^{-x}$. Use a 2nd order Taylor polynomial to give a rational number approximating $f(\frac{1}{2})$.

To second order we have $e^x \approx 1 + x + \frac{x^2}{2}$ so $e^{-x} \approx 1 + (-x) + \frac{(-x)^2}{2} = 1 - x + \frac{x^2}{2}$. Adding the two we find

$$e^x + e^{-x} \approx 2 + x^2\,.$$

At $x = \frac{1}{2}$ this reads

$$e^{1/2} + e^{-1/2} \approx 2\frac{1}{4}\,.$$

[3] f. Show that the error in the approximation is less than $\frac{1}{50}$. You may use the fact that $2 \leq e \leq 3$.

We note that the expansion above is correct to 3rd order (the next term would be $\frac{x^3}{6} + \frac{(-x)^3}{6}$). The error is therefore at most

$$\frac{\left|f^{(4)}(c)\right|}{24}\left(\frac{1}{2}\right)^4$$

for some $0 \leq c \leq \frac{1}{2}$. Now the fourth derivative of e^x is e^x, and of e^{-x} is $(-1)^4 e^{-x} = e^{-x}$ so $f^{(4)}(x) = f(x) = e^x + e^{-x}$. For $0 \leq c \leq \frac{1}{2}$ we then have $f^{(4)}(c) = e^c + e^{-c} \leq \sqrt{e} + e^0 \leq \sqrt{4} + 1 = 3$. It follows that the error is at most

$$\frac{3}{24} \cdot \frac{1}{16} = \frac{1}{8 \cdot 16} = \frac{1}{128} < \frac{1}{50}\,.$$

2 Long-form answers

[6] Find the maximum value of $f(x) = x\sqrt{1-\frac{3}{4}x^2}$ **on the interval** $[0,1]$**.**

- For $|x| < 1$ we have $\frac{3}{4}x^2 \le \frac{3}{4} < 1$ so by the chain rule $\sqrt{1-\frac{3}{4}x^2}$ is differentiable in the open interval $(0,1)$. By the product rule the same holds for f. f is also continuous where defined, so it is continuous in the closed interval. It follow that the maximum of f occurs either at an endpoint or at a critical point.
- We have $f(0) = 0$ and $f(1) = \sqrt{1-\frac{3}{4}} = \frac{1}{2}$.
- We have $f'(x) = \sqrt{1-\frac{3}{4}x^2} + x\frac{1}{2\sqrt{1-\frac{3}{4}x^2}}\left(-\frac{3}{2}x\right) = \frac{1-\frac{3}{4}x^2-\frac{3}{4}x^2}{\sqrt{1-\frac{3}{4}x^2}} = \frac{1-\frac{3}{2}x^2}{\sqrt{1-\frac{3}{4}x^2}}$.It follows that $f'(x_0) = 0$ when $\frac{3}{2}x_0^2 = 1$, that is for $x_0 = \sqrt{\frac{2}{3}}$.
- At the critical point, we have $f(\sqrt{\frac{2}{3}}) = \sqrt{\frac{2}{3}}\sqrt{1-\frac{3}{4}\cdot\frac{2}{3}} = \sqrt{\frac{2}{3}\left(1-\frac{1}{2}\right)} = \frac{1}{\sqrt{3}}$.
- Finally, since $3 < 4$ we have $\sqrt{3} < \sqrt{4}$ so $\frac{1}{\sqrt{3}} > \frac{1}{2} > 0$ so the absolute maximum is $\frac{1}{\sqrt{3}}$.

3 Long-form answers

[6] A point is moving on the curve $y^2 = x^3 - 3$ in such a way that the x-co-ordinate is changing at the rate of $3\frac{\text{units}}{\text{min}}$. How fast is the <u>distance of the point to the origin</u> changing, when the point is at $(2, \sqrt{5})$?

The distance of the point from the origin is

$$D(t) = \sqrt{x^2 + y^2} = \sqrt{x^2 + x^3 - 3}\,.$$

It follows that

$$\begin{aligned}\frac{dD}{dt} &= \frac{1}{2\sqrt{x^2+x^3-3}}\frac{d}{dt}(x^2+x^3-3)\\ &= \frac{2x+3x^2}{2\sqrt{x^2+x^3-3}}\frac{dx}{dt}\,.\end{aligned}$$

When $x = 2$ and $\frac{dx}{dt} = 3\frac{\text{units}}{\text{min}}$ this gives

$$\begin{aligned}\frac{dD}{dt} &= \frac{4+12}{2\sqrt{4+8-3}} \cdot 3\frac{\text{units}}{\text{min}}\\ &= \frac{16}{2\sqrt{9}} \cdot 3\frac{\text{units}}{\text{min}}\\ &= 8\frac{\text{units}}{\text{min}}\,.\end{aligned}$$

4 Long-form answers

Let $f(x) = e^x - e^{-x}$.

[1] Verify that $(f'(x))^2 = 4 + (f(x))^2$.

We have $f'(x) = e^x + e^{-x}$. Since $e^x \cdot e^{-x} = e^{x-x} = 1$ we have $(f(x))^2 = e^{2x} - 2 + e^{-2x}$ and $(f'(x))^2 = e^{2x} + 2 + e^{-2x} = 4 + e^{2x} - 2 + e^{-2x}$.

[3] Show that f has an inverse function.

Three approaches:

1. From the first part we know that $(f'(x))^2 \geq 4$ and in particular it is never zero. Since $f'(0) = 1 + 1 = 2 > 0$ we find that $f'(0) > 0$ everywhere (f' is continuous, so if it took negative and positive values it would vanish). It follows that f is strictly monotone and so has an inverse function.

2. We know that $e^x > 0$ for all x. It follows that $f'(x) = e^x + e^{-x} > 0$ everywhere.

3. Let y be any real number. We will show that that is a unique x such that $f(x) = y$. Consider the equation $t - \frac{1}{t} = y$ for an unkown $t > 0$ (t stands for e^x), equivalently $t^2 - yt - 1 = 0$. This has the two solutions $\frac{y \pm \sqrt{y^2+4}}{2} = \frac{y}{2} \pm \sqrt{\left(\frac{y}{2}\right)^2 + 1}$. Since $\sqrt{\left(\frac{y}{2}\right)^2 + 1} > \left|\frac{y}{2}\right|$ the solution $\frac{y}{2} - \sqrt{\left(\frac{y}{2}\right)^2 + 1}$ is negative. On the other hand, the solution $t = \frac{y}{2} + \sqrt{\left(\frac{y}{2}\right)^2 + 1}$ is positive (even if y is negative). So there is a unique positive value of t such that $t + \frac{1}{t} = y$. Given t, there is a unique x such that $e^x = t$ (that is $x = \ln t$) so indeed for every y there is a unique x such that $f(x) = y$ – in other words, f has an inverse function. That function is

$$x = \ln\left[\frac{y}{2} + \sqrt{\left(\frac{y}{2}\right)^2 + 1}\right].$$

[3] Let $x = g(y)$ be the inverse function. Find a formula for its derivative in terms of x, y.

By the inverse function rule we have

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{e^x + e^{-x}} .$$

If you took the third approach before and found $g(y) = \ln\left[\frac{y}{2} + \sqrt{\left(\frac{y}{2}\right)^2 + 1}\right]$, you would now find

$$\begin{aligned} g'(y) &= \frac{1}{\frac{y}{2} + \sqrt{\left(\frac{y}{2}\right)^2 + 1}} \left[\frac{1}{2} + \frac{y/2}{2\sqrt{(y/2)^2 + 1}}\right] \\ &= \frac{1}{\frac{y}{2} + \sqrt{\left(\frac{y}{2}\right)^2 + 1}} \left[\frac{\sqrt{(y/2)^2 + 1} + y/2}{2\sqrt{(y/2)^2 + 1}}\right] \\ &= \frac{1}{\sqrt{y^2 + 4}} . \end{aligned}$$

[3] Find a formula for $g'(y)$ involving y alone.

From the first part we know that $f'(x) = \sqrt{4 + (f(x))^2} = \sqrt{4 + y^2}$ so

$$g'(y) = \frac{1}{\sqrt{4 + y^2}} .$$