Math 100 §105, Fall Term 2010
Sample midterm Exam

October 4^{th},2010

Student number:

LAST name:

First name:

Instructions

- Do not turn this page over. You will have 45 minutes for the exam.
- You may not use books, notes or electronic devices of any kind.
- Solutions should be written clearly, in complete English sentences, showing all your work.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.

Signature:

1	/18
2	/8
3	/6
4	/8
Total	/40

1 Short-form answers

Show your work and clearly delineate your final answer. Not all problems are of equal difficulty.

[3] a. Evaluate the following limit (or show it does not exist):

$$\lim_{x\to -2}\frac{x^2-4}{x^2+2x+1} = \frac{\lim_{x\to -2}(x^2-4)}{\lim_{x\to -2}(x^2+2x+1)} = \frac{0}{1} = 0$$

where the second equality holds since polynomials are continuous, and the first holds by the quotient rule (since the limit of the denominator is non-zero).

[3] b. Evaluate the following limit (or show it does not exist):

$$\lim_{x\to 0}\frac{e^{3x}-1}{x}$$

Writing $f(x) = e^{3x}$ and noting that $f(0) = 1$ the given limit is simply the derivative $f'(0)$. By the chain rule $f'(x) = 3e^{3x}$ so the given limit is $3e^0 = 3$.

[3] c. Evaluate the following limit (or show it does not exist):

$$\lim_{x\to\infty}\frac{x\cos x}{x^2+1}$$

For any x we have $-1 \le \cos x \le 1$ and $0 < \frac{1}{x^2+1} \le \frac{1}{x^2}$. Thus for x positive we have

$$-\frac{1}{x} = \frac{-x}{x^2} \le \frac{x\cos x}{x^2+1} \le \frac{x}{x^2} = \frac{1}{x}\,.$$

Since $\lim_{x\to\infty}\frac{1}{x} = 0$ the same holds for $\lim_{x\to\infty} -\frac{1}{x}$, so by the squeeze rule,

$$\lim_{x\to\infty}\frac{x\cos x}{x^2+1} = 0\,.$$

[3] d. Differentiate the following function:

$$\tan(e^{x/2})$$

Applying the chain rule twice,

$$\frac{d}{dx}\left(\tan(e^{x/2})\right) = \left(1+\tan^2(e^{x/2})\right)\cdot\frac{d}{dx}\left(e^{x/2}\right) = \left(1+\tan^2(e^{x/2})\right)\frac{1}{2}e^{x/2}$$

where we have used $\frac{d}{dx}(\tan x) = 1+\tan^2 x$ and $\frac{d}{dx}e^x = e^x$.

[3] e. Given $f(1) = 1$, $f'(1) = 2$, $g(1) = 3$, $g'(1) = 4$ evaluate $h'(1)$ where

$$h(x) = \frac{xg^2(x)}{f(x)}\,.$$

By the product rule, $\left(xg^2(x)\right)' = g^2(x) + 2xg(x)g'(x)$. By the quotient rule it follows that

$$h'(x) = \frac{g^2(x) + 2xg(x)g'(x)}{f(x)} - \frac{xg^2(x)f'(x)}{f^2(x)}$$

Evaluting at $x = 1$ we find:

$$h'(1) = \frac{3^2 + 2\cdot 1\cdot 3\cdot 4}{1} - \frac{1\cdot 3^2\cdot 2}{1^2} = 33 - 18 = 15\,.$$

[3] f. Evaluate the following limit (or show it does not exist):

$$\lim_{x\to 0}\frac{\sqrt{1-\cos x}}{x}$$

We have $\frac{\sqrt{1-\cos x}}{x} = \frac{\sqrt{1-\cos x}}{x}\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}} = \frac{\sqrt{1-\cos^2 x}}{x\sqrt{1+\cos x}} = \frac{|\sin x|}{x\sqrt{1+\cos x}}$. since $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\frac{1}{\sqrt{1+\cos x}}$ is continous at $x = 0$ we use the product rule to find:

$$\lim_{x\to 0^+}\frac{\sqrt{1-\cos x}}{x} = \lim_{x\to 0^+}\frac{\sin x}{x}\cdot\lim_{x\to 0^+}\frac{1}{\sqrt{1+\cos x}} = 1\cdot\frac{1}{\sqrt{2}}$$

and

$$\lim_{x\to 0^-}\frac{\sqrt{1-\cos x}}{x} = \lim_{x\to 0^+}\frac{-\sin x}{x}\cdot\lim_{x\to 0^+}\frac{1}{\sqrt{1+\cos x}} = -1\cdot\frac{1}{\sqrt{2}}$$

It follows that the limit does not exist.

2 Long-form answers

[4] a. Let $f(x) = \frac{x}{x-1}$. Find $f'(x)$ *using the definition of the derivative.* No marks will be given for use of differentiation rules.

Let $x \neq 1$ so that f is defined. We need to evalute the limit

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{1}{h}\left(\frac{x+h}{x+h-1} - \frac{x}{x-1}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{(x+h)(x-1) - x(x+h-1)}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2+hx-x-h-x^2-hx+x}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{(x-1)(x+h-1)}\right) \\
&= \lim_{h\to 0} \frac{-1}{x-1}\cdot\frac{1}{x+h-1}
\end{aligned}$$

Since $\frac{-1}{x-1}$ is a constant the last limit equals

$$-\frac{1}{x-1}\lim_{h\to 0}\frac{1}{x+h-1}\,.$$

By the quotient rule this equals

$$-\frac{1}{x-1}\frac{1}{\lim_{h\to 0}(x+h-1)} = -\frac{1}{x-1}\cdot\frac{1}{x-1} = -\frac{1}{(x-1)^2}\,.$$

[4] b. Show that the equation $\cos x = x$ has a solution.

Consider the function $f(x) = \cos x - x$. It is continuous, being the difference of continuous functions. Also, we have $f(0) = 1 - 0 = 1 > 0$ and $f(\frac{\pi}{2}) = 0 - \frac{\pi}{2} = -\frac{\pi}{2} < 0$. By the intermediate value theorem there is x, $0 < x < \frac{\pi}{2}$ such that $f(x) = 0$, in other words such that $\cos x = x$.

3 Long-form answers

A reaction occurs at the rate $R(x) = Ax^3 e^{-x/E}$ where x is the energy of the incoming particles, E is a constant energy scale, and A is an overall constant.

[4] a. Find the range of energies $x \geq 0$ for which a small increase in the energy will increase the rate of the reaction.

[2] b. Find the range of energies $x \geq 0$ for which a small increase in the energy will decrease the rate of the reaction. Your answers may depend on the constants A and E. Don't forget to justify them!

By the product rule we have $R'(x) = 3Ax^2 e^{-x/E} - Ax^3 \frac{1}{E} e^{-x/E} = Ax^2 e^{-x/E} \cdot \left(3 - \frac{x}{E}\right)$ .Since $Ax^2 e^{-x/E}$ is always positive, $R'(x) > 0$ when $3 - \frac{x}{E} > 0$ and $R'(x) < 0$ when $3 - \frac{x}{E} < 0$. In other words, $R'(x)$ is increasing for $0 \leq x < 3E$ and decreasing for $x > 3E$.

4 Long-form answers

[7] a. Write down two equations in the two unknowns a, b expressing the statement: "the line tangent to $y = \sqrt{x} - \frac{1}{2}$ at the point where $x = a$ is also tangent to $y = x^2$ at the point where $x = b$".
Since the derivative of $y = \sqrt{x} - 1$ is $\frac{1}{2\sqrt{x}}$, the line tangent to that curve at $x = a$ has the slope $\frac{1}{2\sqrt{a}}$ and hence the equation

$$y = \frac{1}{2\sqrt{a}}(x - a) + \left(\sqrt{a} - \frac{1}{2}\right)$$

that is

$$y = \frac{1}{2\sqrt{a}}x + \left(\frac{\sqrt{a}}{2} - \frac{1}{2}\right).$$

Since the derivative of $y = x^2$ is $2x$, the line tangent to that curve at $x = b$ has the equation

$$y = 2b(x - b) + b^2$$

that is

$$y = 2b \cdot x - b^2.$$

Now if both equations describe the same line we must have

$$\begin{cases} 2b = \frac{1}{2\sqrt{a}} \\ \frac{\sqrt{a}}{2} - \frac{1}{2} = -b^2 \end{cases}$$

[1] b. Solve the system of equations you have written down.
Multiplying the second equation by $8b$ we find:

$$4\sqrt{a}b - 4b = -8b^3.$$

From the first equation we have $4\sqrt{a}b = 1$ so

$$(2b)^3 - 2(2b) + 1 = 0.$$

In terms of $c = 2b$ this reads

$$c^3 - 2c + 1 = 0.$$

Given any solution c to the last equation, it is clear that setting $b = \frac{c}{2}$ and $a = \left(\frac{1}{2c}\right)^2$ will give a solution to the original system as long as b comes out positive (so that $b = \frac{1}{4\sqrt{a}}$ rather than its negative), so it is enough to solve the last equation. By inspection we find the solution $c = 1$; since $c^3 - 2c + 1 = (c - 1)(c^2 + c - 1)$ there are also the solutions $\frac{-1 \pm \sqrt{5}}{2}$. As we said c must be non-negative since $c = 2b = \frac{1}{2\sqrt{a}}$ so the two possibilities are $c = 1$ and $c = \frac{\sqrt{5}-1}{2}$ which translate to $\left(a = \frac{1}{4}, b = \frac{1}{2}\right)$ and $\left(a = \frac{1}{\left(\sqrt{5}-1\right)^2}, b = \frac{\sqrt{5}-1}{4}\right)$.