Mid-term II, (Monday November 15, 2004)
Announcement

Topics covered
* Exponential, *l*n and inverse trigonometric functions.
* Growth and decay problems, Differential Equations, Newton's law of cooling.
* Implicit differentiation, related rates.
* Newton's method, linear approximation (= tangent line approximation) .
* The mean value theorem and its uses ,
e.g. discussion of increasing and decreasing functions, etc. .
* Local classification of critical points via first derivative sign test.

Calculators are NOT allowed.

Added Office hours :
* Friday November 12, 11:15am - 1:20pm
* Monday November 15 : 11:15am - 1:20pm

Suggested Review Problems

§ 3.8,	(P.187)	# 55, 58
§ 3.10	(P.209)	# 26, 27, 39
§4.3	(P.236)	# 34, 49, 55, 57
§4.4	(P.244)	# 9, 19, 23, 37, 43
	(P.297)	# 5, 15 (include error estimate)
	(P.298)	# 21, 23, 28, 30, 33, 34, 41
§6.8	(P.475)	# 30, #64 b(ii), 72
§8.4	(P.589)	# 39, 42, 43
	(P.620)	#32, 33, 34

Mid-term II, Nov 15, 2004
Comments / Solutions on some problems

Question 1

(A) This is a straight-forward question on linear approximation (= tangent line approximation).

Compare with
(i) home-work assignment VII, p.225, #27
(ii) text-book example 2, p.220.
(iii) class example on the 4 millionth root of 0.92

(B) The differential equation $\frac{dI}{dx} = kI$ where k is a constant, should be familiar.

Compare with

(i) home-work assignment V, p.557, #2, #23, and assignment VI, p.576, #32.

(ii) text-book example p.552, equation (16); and

(iii) class example about sugar dissolving in water.

(C) This is a standard application of Newton's method. Compare with

(i) Math 100 final exam, Dec.1999, Question 10, and similar questions from other years,

(ii) text-book example 4, p.204,

(iii) class example on solving $x^3 - 2x - 5 = 0$

(D) This is another standard problem. The local minimum point on the graph is (2, -11). You can use either the first derivative sign test or the second derivative test to see that x = 2 gives local minimum.

(E) Use implicit differentiation to find the tangent slope y' at (0,1). The normal slope is then - 1/y' , and from this the equation of the normal line at (0,1) can be easily found.

Question 2

To solve $\frac{dy}{dt} = -2r - 2y$ where r is a constant, write the differential equation as $\frac{dy}{dt} = -2\,(y + r)$ and then as $\frac{d}{dt}(y + r) = -2\,(y + r)$.

Hence $y + r = C\,e^{-2t}$ where C is some constant whose value can be determined because we are given y(0). Figure out this value and get $y = C\,e^{-2t} - r$ to be the answer. Compare with

(i) home-work assignment VI, p.576, #29, #32,

(ii) text-book example 4, p.572,

(iii) class example on "determining the time of death" in connection with a murder case.

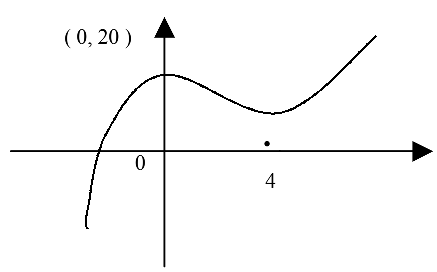
Part (b). From $y = e^{-2t} - r$ we get $\lim_{t\to\infty} y = -r$, because $e^{-2t} \to 0$ as $t \to \infty$

Part (c). Since $\frac{dy}{dt} = -2\,C\,e^{-2t}$, the above gives $\lim_{t\to\infty} \left(\frac{dy}{dt}\right) = 0$, a very simple answer!

Question 3

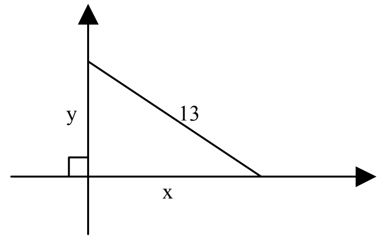
As pointed out several times in class, curve sketching problems are easy. Follow the procedure outlined in the text, p.276 whenever possible. Use tables to summarize all relevant information, such as the signs of f '(x), f "(x) over various intervals. For $f(x) = \frac{1}{20}x^5 - \frac{1}{4}x^4 + 20$,

the graph looks approximately like the one shown .



Question 4

This is a related rate problem .frequently solvable by implicit differentiation.



Since $x^2 + y^2 = 13^2$,

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

When $y = 5$, $x = \sqrt{13^2 - 5^2} = 12$ and substitution gives $2 \times 12 \times 0.25 + 2 \times 12 \frac{dy}{dt} = 0$.

Therefore $\frac{dy}{dt} = -0.6$, which means that the top of the ladder is sliding down at 0.6 m/s.

Question 5

(a) The statement of the Mean Value Theorem has to be very precise. If you score less than 4 points in this part, it means your statement has some flaw in it. Consult class notes or text (p.229) to see what the precise statement should be, and understand where you have gone wrong.

(b) Let $f(x) = \tan^{-1} x$, a differentiable function with $A = (a, f(a))$, $B = (b, f(b))$ on its graph.

The slope m of $\overline{AB}$ is $m = \frac{f(b) - f(a)}{b - a}$. By the Mean Value Theorem, there is a number c between a, b such that $m = f'(c)$. But $f'(x) = \frac{1}{1+x^2}$, so $m = \frac{1}{1+c^2}$. Since $1 + c^2$ is positive, $m > 0$. Since $1 + c^2 = 1 + (\text{non-negative amount}) \geq 1$, one has $\frac{1}{1+c^2} \leq \frac{1}{1} = 1$.

Hence $0 < m \leq 1$. (Please pay special attention to the fine logical flow in this argument , an attribute to good mathematics writing).

(c) Let $g(x) = x - \sin x$, to be considered over the interval $[0, \infty)$. Note $g(0) = 0$, $g'(x) = 1 - \cos x \geq 0$. This, by Mean Value Theorem, ensures that g (x) is an increasing function on $[0, \infty)$. In particular, for $x \geq 0$, $g(x) \geq g(0)$, which translates into $x \geq \sin x$. End of proof.

(As remarked in class earlier, one of the uses of the Mean Value Theorem is to establish some inequalities, such as the one shown here).