Sample Midterm 2 for Math 105

1. Find the derivative of the function

$$f(x) = x^2 \int_3^x t \sin\left(\frac{\pi t}{6}\right) dt$$

at the point $x = 3$.

2. Use Simpson's rule to approximate

$$\int_1^2 \ln x \, dx$$

with $n = 4$ subintervals. Find a bound on the error. **No need to simplify your answers!**

3. (a) Find the indefinite integral

$$\int \sin^3(x) \cos^{10}(x) \, dx.$$

(b) Obtain the partial fraction decomposition of the function

$$\frac{x-7}{x^2 - x - 12}.$$

4. (a) Find the definite integral

$$\int_0^{\pi/4} \sec^2 x \, dx.$$

(b) Evaluate $\int_1^2 f(3x) f'(3x) \, dx$, where $f'(x)$ is the derivative of $f(x)$, $f(6) = 0$ and $f(3) = 1$.

5. Solve the initial value problem

$$e^{-t}y' = \frac{t}{y}, \qquad y(0) = -5.$$

6. Evaluate the definite integral:

$$\int_0^{\frac{\ln(\sqrt{3})}{2}} \frac{e^{2t}}{(1+e^{4t})^{\frac{3}{2}}}\, dt.$$

7. Evaluate the following limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{6(k-1)^2}{n^3}\sqrt{1+2\frac{(k-1)^3}{n^3}}$$

Solutions to Sample Midterm 2 for Math 105

1. $f'(x) = 2x\int_3^x t\sin\left(\frac{\pi t}{6}\right)dt + x^2 \cdot x\sin\left(\frac{\pi x}{6}\right) \Rightarrow f'(3) = 0 + 3^3\sin\left(\frac{\pi}{2}\right)$
$= 27.$

2. $S(4) = \left[\ln 1 + 4\ln\left(\frac{5}{4}\right) + 2\ln\left(\frac{3}{2}\right) + 4\ln\left(\frac{7}{4}\right) + \ln 2\right]\frac{1/4}{3}$

$f(x) = \ln x \Rightarrow f^{(4)}(x) = -\frac{6}{x^3} \Rightarrow |f^{(4)}(x)| \le 6$ for x in $[1,2]$. Hence, the error bound $E_{S(4)} \le \frac{K\cdot(2-1)}{180}(\Delta x)^4 = \frac{6}{180}\left(\frac{1}{4}\right)^4.$

3. (a) $\int \sin^3 x\cos^{10}x\,dx = \int \sin x\left(1-\cos^2 x\right)\cos^{10}x\,dx \underset{\substack{u=\cos x\\ du=-\sin x\,dx}}{=} \int (1-u^2)u^{10}(-du)$

$= -\int (u^{10} - u^{12})\,du = -\frac{u^{11}}{11} + \frac{u^{13}}{13} + C = -\frac{\cos^{11}x}{11} + \frac{\cos^{13}x}{13} + C.$

(b) $\frac{x-7}{x^2-x-12} = \frac{x-7}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}$, $\quad x-7 = A(x+3) + B(x-4)$

$x=4 \Rightarrow -3 = 7A \Rightarrow A = -3/7$; $\quad x=-3 \Rightarrow -10 = -7B \Rightarrow B = 10/7.$

$\frac{x-7}{x^2-x-12} = \frac{(-3/7)}{x-4} + \frac{10/7}{x+3}$

4. (a).

$$\int_0^{\pi/2} \sec^2 x\,dx =$$

$$= \lim_{b\to(\frac{\pi}{2})^-} \int_0^b \sec^2 x\,dx = \lim_{b\to(\pi/2)^-} \tan x \Big|_0^b = \lim_{b\to(\pi/2)^-} (\tan b - 0) = \infty.$$

Hence, the integral is divigent.

(b)

$$\int_1^2 f(3x) f'(3x)\,dx \underset{\substack{u=f(3x),\ v'=f'(3x)\\ u'=3f'(3x),\ v=\frac{1}{3}f(3x)}}{=\!=\!=} \left(\frac{1}{3} f(3x)^2\right)\Big|_1^2 - \int_1^2 f'(3x) f(3x)\,dx$$

$$2\int_1^2 f(3x) f'(3x)\,dx = \frac{1}{3}\left(f(6)^2 - f(3)^2\right) = \frac{1}{3}(0-1) = -\frac{1}{3}$$

$$\int_1^2 f(3x) f'(3x)\,dx = -\frac{1}{6}.$$

5.

$$\frac{dy}{dt} = \frac{te^t}{y} \Rightarrow y\,dy = te^t\,dt \Rightarrow \frac{y^2}{2} = \int y\,dy = \int te^t\,dt =$$

$$\underset{\substack{u=t,\ v'=e^t\\ u'=1,\ v=e^t}}{=\!=\!=} te^t - \int e^t\,dt = te^t - e^t + C. \qquad \frac{(-5)^2}{2} = -e^0 + C \Rightarrow C = \frac{27}{2}.$$

Hence, $\frac{y^2}{2} = te^t - e^t + \frac{27}{2} \Rightarrow y^2 = 2te^t - 2e^t + 27$

$$y = -\sqrt{2te^t - 2e^t + 27}.$$

6. $$\int_0^{\frac{\ln(\sqrt{3})}{2}} \frac{e^{2t}}{(1+e^{4t})^{3/2}}\,dt \underset{\substack{u=e^{2t}\\ du=2e^{2t}dt}}{=} \int_1^{\sqrt{3}} \frac{du/2}{(1+u^2)^{3/2}} \underset{\substack{u=\tan\theta\\ du=\sec^2\theta\,d\theta}}{=} \frac{1}{2}\int_{\pi/4}^{\pi/3} \frac{\sec^2\theta\,d\theta}{\sec^3\theta}$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/3} \frac{d\theta}{\sec\theta} = \frac{1}{2}\int_{\pi/4}^{\pi/3} \cos\theta\,d\theta = \frac{1}{2}\sin\theta\Big|_{\pi/4}^{\pi/3} = \frac{1}{2}\left(\sin\frac{\pi}{3} - \sin\frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right).$$

7. The limit $= \int_0^1 6x^2\sqrt{1+2x^3}\,dx$

$$\underset{\substack{u=1+2x^3\\ du=6x^2dx}}{=} \int_1^3 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_1^3 = \frac{2}{3}\left(3^{3/2}-1\right).$$