Sample Midterm I

[6] 1. Let Q be the plane described by the equation $x + y + 2z = 2$.

(a) [3] Find the point at which Q intersects the y-axis.

(b) [3] Find the value of a constant c such that Q is orthogonal to the plane $R : 2x + cy - 3z = 0$.

[7] 2. Let $z = f(x, y) = 5y - x^2$.

(a) [4] Explain briefly what the shape of the level curves of the function $z = f(x, y)$ is, such as lines, circles, and parabolas. Sketch the level curves $f(x, y) = z_0$ with $z_0 = 0$ and $z_0 = -5$.

(b) [3] Is it possible that both the point $(1, 2)$ and the point $(-1, 3)$ are on the same level curve of the function $z = f(x, y)$? Please justify your answer.

[6] 3. The function $f(x,y)$ obeys

$$f(x,y) + \sin\big(f(x,y)\big) = 2x + 4xy \quad \text{and } f(0,0) = 0$$

(a) [3] Find $\dfrac{\partial f}{\partial x}(0,0)$.

(b) [3] Find $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.

[5] 4. Find an equation of the plane that passes through the point $P(a, b, c)$ and is perpendicular to the line connecting the point P and the origin. Your answer should be in terms of the constants a, b, and c.

[10] 5. Let $f(x, y) = 2x^3 - x^2y + y^2 - 5y$.

(a) [5] Find all critical points of $f(x, y)$.

(b) [5] Determine whether each critical point corresponds to a local maximum, local minimum, or saddle point.

[10] 6. Let R be the set $\{(x, y) \mid x^2 + y^2 \leq 1\}$.

(a) [5] Use Lagrange multipliers to find the maximum and minimum values of $2x^2 + y^2 - y$ on the boundary of the set R. A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.

(b) [5] Find the maximum and minimum values of $2x^2 + y^2 - y$ on the set R.

Solutions to Sample Midterm 1

1. (a) Let $x=0$ and $z=0$. We get $0+y+2\cdot 0=2$ or $y=2$. Hence, the point is $(0,2,0)$.

(b) Since Q and R are orthogonal, their respective normal vectors are orthogonal. Thus, we get

$$0=\langle 1,1,2\rangle\cdot\langle 2,c,-3\rangle=2+c-6$$

or $c=4$. //

2. (a) Any level curve of $z=f(x,y)$ has an equation $5y-x^2=z_0$ or $y=\frac{1}{5}x^2+\frac{z_0}{5}$ for some constant z_0. Hence, all level curves of $z=f(x,y)$ are parabolas.

The level curves $f(x,y)=z_0$ with $z_0=0$ and $z_0=-5$ are given in Fig. 1.

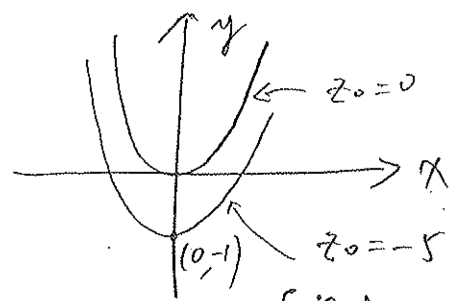

Fig. 1

(b) No. In fact, if $(1,2)$ is on a level curve, then the equation of the level curve must be $5y-x^2=5\cdot 2-1^2$ or $5y-x^2=9$. Since $5\cdot 3-(-1)^2=14\neq 9$, it is impossible that $(-1,3)$ is also on the level curve: $5y-x^2=9$.

3.

Solution. (a) Applying $\frac{\partial}{\partial x}$ to $f(x,y) + \sin\big(f(x,y)\big) = 2x + 4xy$ gives

$$f_x(x,y) + f_x(x,y)\cos\big(f(x,y)\big) = 2 + 4y$$

Then setting $x = y = 0$, and using that $\cos\big(f(0,0)\big) = \cos 0 = 1$, gives

$$f_x(0,0) + f_x(0,0)\cos\big(f(0,0)\big) = 2 \implies 2f_x(0,0) = 2 \implies \boxed{f_x(0,0) = 1}$$

(b) Applying $\frac{\partial}{\partial y}$ to $f_x(x,y) + f_x(x,y)\cos\big(f(x,y)\big) = 2 + 4y$ gives

$$f_{xy}(x,y) + f_{xy}(x,y)\cos\big(f(x,y)\big) - f_x(x,y)\, f_y(x,y)\sin\big(f(x,y)\big) = 4$$

Then setting $x = y = 0$, and using that $\cos\big(f(0,0)\big) = \cos 0 = 1$ and $\sin\big(f(0,0)\big) = \sin 0 = 0$, gives

$$\begin{aligned}&f_{xy}(0,0) + f_{xy}(0,0)\cos\big(f(0,0)\big) - f_x(0,0)\, f_y(0,0)\sin\big(f(0,0)\big) = 4\\ &\implies 2f_{xy}(0,0) = 4\\ &\implies \boxed{f_{xy}(0,0) = 2}\end{aligned}$$

4. $\overrightarrow{OP} = \langle a, b, c\rangle$ is a normal vector to the plane. Hence, an equation of the plane is

$$a(x-a) + b(y-b) + c(z-c) = 0$$

or $ax + by + cz = a^2 + b^2 + c^2$.

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5. (a) Compute f_x and f_y:

$$f_x = 6x^2 - 2xy = 2x(3x - y), \quad f_y = -x^2 + 2y - 5.$$

We see that both are defined for all x and y. Therefore the only critical points are where $f_x = f_y = 0$. By letting $f_x = 0$, we get $x = 0$ or $y = 3x$. We consider these conditions one at a time

Case 1 If $x = 0$, then setting $f_y = 0$ gives us $2y = 5$. Therefore the point $(0, \frac{5}{2})$ is a critical point of f.

Case 2 If $y = 3x$, then $f_y = 0$ gives $-x^2 + 6x - 5 = 0$. We can factor this as $-(x-1)(x-5) = 0$. So the points $(1, 3)$ and $(5, 15)$ are also critical points.

(b) $f_{xx} = 12x - 2y$, $f_{xy} = -2x$, $f_{yy} = 2$,

$D(x, y) = f_{xx} f_{yy} - f_{xy}^2$

Critical points	f_{xx}	f_{yy}	f_{xy}	$D(x,y)$	Conclusion
$(0, \frac{5}{2})$	-5	2	0	$-$	saddle point
$(1, 3)$	6	2	-2	$+$	local min.
$(5, 15)$	30	2	-10	$-$	saddle point

6. (a) To find the maximum and minimum values of f on the boundary, we must find the max and min of $f(x,y) = 2x^2 + y^2 - y$ subject to the constraint that $g(x,y) = x^2 + y^2 - 1 = 0$. We use the method of Lagrange multipliers. The extremals obey

$$0 = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 4x - 2\lambda x = 2x(2 - \lambda) \quad (1)$$
$$0 = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 2y - 1 - 2\lambda y \quad (2)$$
$$0 = x^2 + y^2 - 1 \quad (3)$$

Equation (1) implies that either $x = 0$ or $\lambda = 2$. If $x = 0$, equation (3) gives $y = \pm 1$. If $\lambda = 2$, equation (2) gives $2y - 1 - 4y = 0$ or $y = -\frac{1}{2}$. Then equation (3) gives $x = \pm\sqrt{\frac{3}{4}}$. Hence, all candidates for max. and min are given in the following table:

Candidate	f
$(\frac{\sqrt{3}}{2}, -\frac{1}{2})$	$9/4$
$(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$	$9/4$
$(0, 1)$	0
$(0, -1)$	2

Thus, the max and min. of $2x^2+y^2-y$ on the boundary of R are $\frac{9}{4}$ and 0, respectively.

(b) For $f(x,y) = 2x^2+y^2-y$, we have $f_x = 4x$ and $f_y = 2y-1$. Hence, the only critical point is $(0, \frac{1}{2})$. Using $f(0, \frac{1}{2}) = -\frac{1}{4}$ and the table in (a), we know that the max. and min. of $2x^2+y^2-y$ on the set R are $\frac{9}{4}$ and $-\frac{1}{4}$, respectively.