Problem 9. Are the following functions pdf's?

(a) $f(x) = \begin{cases} 12x^2(x-1) & \text{if } 0 < x < 1 \\ 0 & \text{if } elsewhere \end{cases}$

(b) $f(x) = \begin{cases} 1-|x| & \text{if } |x| \le 1 \\ 0 & \text{if } elsewhere \end{cases}$

(c) $f(x) = \begin{cases} \frac{\pi}{2}\cos(\pi x) & \text{if } |x| < \frac{1}{2} \\ 0 & \text{if } elsewhere \end{cases}$

(d) $f(x) = \begin{cases} \frac{1}{2} & \text{if } |x| \le 2 \\ 0 & \text{if } elsewhere \end{cases}$

Solution 9:
(a) No; $f(x) < 0$.
(b) Yes.
(c) Yes.
(d) No; $\int_{-\infty}^{\infty} f(x)dx = 2$.

Problem 10. For the functions in Problem 9 that you found to be probability density functions, find the corresponding cumulative distribution functions.

Solution 10:

(b) $F(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{x^2}{2} + x + \frac{1}{2} & \text{if } -1 \le x < 0 \\ -\frac{x^2}{2} + x + \frac{1}{2} & \text{if } 0 \le x < 1 \\ 1 & \text{if } x > 1 \end{cases}$

(c) $F(x) = \begin{cases} 0 & \text{if } x \le -\frac{1}{2} \\ \frac{1}{2}(1+\sin(\pi x)) & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases}$

Problem 11. Are the following functions cdf's?

(a) $F(x) = \begin{cases} 0 & \text{if } x < -\frac{\pi}{2} \\ \frac{\sin(x-\pi/2)+1}{2} & \text{if } |x| \le \frac{\pi}{2} \\ 1 & \text{if } x > \frac{\pi}{2} \end{cases}$

(b) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1-\cos(x)}{2} & \text{if } 0 \le x \le \pi \\ 1 & \text{if } x > \pi \end{cases}$

(c) $F(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1+x & \text{if } -1 \le x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$

(d) $F(x) = \arctan(x) + \frac{\pi}{2}$

Solution 11:
(a) No; $F(x)$ is not a nondecreasing function.
(b) Yes.
(c) No; $F(x)$ is not a nondecreasing function.
(d) No; $\lim_{x\to\infty} F(x) = \pi$.

Problem 12. For the functions in Problem 11 that you found to be cumulative distribution functions, find the corresponding probability density functions.

Solution 12:
(b) On $0 \le x \le \pi$, $f(x) = \frac{1}{2}\sin(x)$.

Problem 13. Show that $e^{-x}e^{-e^{-x}}$ on $x \in \mathbb{R}$ is a pdf.

Solution 13: $e^{-x}e^{-e^{-x}} \ge 0$ for all $x \in \mathbb{R}$; $\int_{-\infty}^{\infty} e^{-x}e^{-e^{-x}}dx = -\int_{\infty}^{0} e^{-u}du = 1$.

Problem 14. What is the cdf of the density function $\frac{1}{\pi(1+x^2)}$?

Solution 14: $\int_{-\infty}^{x} \frac{1}{\pi(1+t^2)}dt = \frac{1}{\pi}\arctan(x) + \frac{1}{2}$.

Problem 15. Show that $p(x) = \frac{e^{-x}}{(1+e^{-x})^2}$ on $x \in \mathbb{R}$ is a pdf.

Solution 15: $p(x) \ge 0$ for all $x \in \mathbb{R}$; $\int_{-\infty}^{\infty} \frac{e^{-x}}{(1+e^{-x})^2}dx = -\int_{\infty}^{1} \frac{du}{u^2} = 1$.

Problem 16. Show that $f(x) = \begin{cases} 1 - e^{-x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$ is a cdf.

Solution 16: $\lim_{x\to-\infty} f(x) = 0$; $\lim_{x\to\infty} f(x) = 1$. $f'(x) = e^{-x}$ on $x \ge 0$ which is nonnegative; thus, $f(x)$ is nondecreasing.

Problem 17. Find the constant k that makes the following functions pdf's.

(a) $p(x) = k\sin(x)$, $0 < x < \pi$

(b) $p(x) = kx^2(x-1)^2$, $0 < x < 1$

(c) $p(x) = kx(1-x)^3$, $0 < x < 1$

(d) $p(x) = k,\ -1 \leq x \leq 3$

(e) $p(x) = kx^3 e^{-\frac{x}{2}},\ x \geq 0.$

Solution 17:
(a) $k = \frac{1}{2}$; (b) $k = 30$; (c) $k = 20$; (d) $k = \frac{1}{4}$; (e) $k = \frac{1}{96}$, (use integration by-parts).

Problem 18. For the PDF's in Problem 17, compute the expectations, variances and standard deviations of their associated random variables.

Solution 18:
integration exercises