Solutions to Math 105 Midterm 2 (Version II)

(Feb. 4, 2020)

1. (a) $\langle 3,-1,-c\rangle \cdot \langle 3,1,-1\rangle = 0 \Rightarrow 9-1+c=0 \Rightarrow c=-8.$

$\boxed{\text{Answer: } -8}$

(b) $5(x+3)-(y-2)-2(z-1)=0$

Answer: $5(x+3)-(y-2)-2(z-1)=0$
or
$5x-y-2z=-19$

(c) $f_x(x,y)=2xy^3,\ f_{xy}=6xy^2,\ f_{yx}(3,2)=6\cdot 3\cdot 2^2=72$

$\boxed{\text{Answer: } 72}$

(d) $f_y(2.6,1.8) \approx \dfrac{f_y(2.6,1.7)-f_y(2.6,1.8)}{1.7-1.8} = \dfrac{2.8905-2.9028}{-0.1}$

Answer: $\dfrac{2.8905-2.9028}{-0.1}.$

2. (a) [2] The domain of $f(x,y)$ is given by

$$\{(x,y):\ x^2+\frac{y^2}{2}\le 3\}.$$

(b) [3] Any level curve of $z=f(x,y)$ is given by

$f(x,y)=c$ for some constant c,

or $\sqrt{3-x^2-\frac{y^2}{2}}=c$.

- If $c<0$ or $c>\sqrt{3}$, the level curve does not exist.
- If $c=\sqrt{3}$, the level curve is the point $(0,0)$
- If $0\le c<\sqrt{3}$, the level curve is an ellipse.

(c) [4] $\frac{\partial z}{\partial y}=\frac{1}{2}\left(3-x^2-\frac{y^2}{2}\right)^{-1/2}(-y)=-\frac{y}{2}\left(3-x^2-\frac{y^2}{2}\right)^{-1/2}$.

Hence, $\frac{\partial z}{\partial y}\left(\frac{1}{\sqrt{2}},1\right)=-\frac{1}{2}\left(3-\frac{1}{2}-\frac{1}{2}\right)^{-1/2}=-\frac{1}{2}\left(2^{-1/2}\right)$.

3. (a) [3] $f_{xx} = (y^2-8y)\left[-e^{-x}+(1-x)e^{-x}(-1)\right] = (y^2-8y)e^{-x}(x-2)$.

$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left((y^2-8y)(1-x)e^{-x}\right)$

$= (1-x)e^{-x}\cdot\frac{\partial}{\partial y}(y^2-8y) = (1-x)e^{-x}(2y-8)$

$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left((2y-8)xe^{-x}\right) = 2xe^{-x}$

(b) [5] $D(x,y) = (y^2-8y)(2x)(x-2)e^{-2x} - (2y-8)^2(1-x)^2e^{-2x}$

$D(0,0) = -(0-8)^2 1^2 e^0 = -64 < 0$

$D(0,8) = -(16-8)^2 1^2 \cdot e^0 = -64 < 0$

$D(1,4) = (16-32)\cdot 2(-1)e^{-2} - 0 > 0$

$f_{xx}(1,4) = (16-32)e^{-1}(-1) > 0$

Hence, f has a saddle point at $(0,0)$ and $(0,8)$,

f has a local minimum at $(1,4)$.

4. (a) [3]

$$\frac{\partial}{\partial x}(x^3 - x + y^3 z + z^3) = \frac{\partial}{\partial x}(1)$$

$$3x^2 - 1 + y^3 \cdot \frac{\partial z}{\partial x} + 3z^2 \cdot \frac{\partial z}{\partial x} = 0$$

$$(y^3 + 3z^2)\frac{\partial z}{\partial x} = 1 - 3x^2 \quad (1)$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{1-3x^2}{y^3+3z^2}. \quad (2)$$

(b) [2] If $(x,y) = (1,0)$, then $z^3 = 1$. Hence, $z = 1$.

It follows from (2) that

$$\frac{\partial z}{\partial x}(1,0) = \frac{1-3\cdot 1^2}{0^3+3\cdot 1^2} = -\frac{2}{3}.$$

(c) [4]

$$\frac{\partial}{\partial y}(x^3 - x + y^3 z + z^3) = \frac{\partial}{\partial y}(1)$$

$$3y^2 \cdot z + y^3 \cdot \frac{\partial z}{\partial y} + 3z^2 \cdot \frac{\partial z}{\partial y} = 0$$

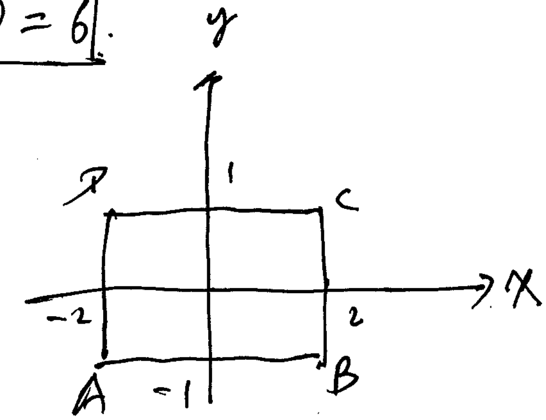
$$(y^3 + 3z^2)\frac{\partial z}{\partial y} = -3y^2 z \quad (3)$$

If $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$, then (1) and (3) give

$$\begin{cases} 1 - 3x^2 = 0 & (4) \\ -3y^2 z = 0 & (5). \end{cases}$$

Since $x^3 - x + y^3 z + z^3 = 1$ -- (6), we get that $z \neq 0$ and $(\pm\frac{1}{\sqrt{3}}, 0)$ are two critical points

5. [8] First, we find the values of f at all critical points in the interior of R. We have $f_x = -2x$, $f_y = -8y$, so $(0,0)$ is the only critical point in R and $\boxed{f(0,0) = 6}$.



Next, find the minimum and maximum values of f on the boundary of R.

On the sides AB and DC, $y = \pm 1$, $-2 \le x \le 2$, we have $f(x, \pm 1) = 6 - x^2 - 4 = 2 - x^2$. $\frac{df}{dx} = -2x = 0$ $\Rightarrow x = 0$. $\boxed{f(0, \pm 1) = 2}$, $\boxed{f(\pm 2, \pm 1) = -2}$

On the sides BC and DA, $x = \pm 2$, $-1 \le y \le 1$, we have $f(\pm 2, y) = 6 - 4 - 4y^2 = 2 - 4y^2$.

$\frac{df}{dy} = -8y = 0 \Rightarrow y = 0$, $f(\pm 2, 0) = 2$, $f(\pm 2, \pm 1) = -2$.

Therefore, the maximum value of f on R is 6, and the minimum value of f on R is -2.

5-5

6.(4) $f_x = x^2 + ay^2 + bx + cy + 6$

$f_{xx} = 2x + b$ we have $\begin{cases} f_{xx}(2,1) = 0 \\ f_{xy}(2,1) = 0 \end{cases}$ or

$f_{xy} = 2ay + c$

$$\begin{cases} 2 \cdot 2 + b = 0 \\ 2a + c = 0 \end{cases} \Rightarrow \begin{cases} \boxed{b = -4} \\ c = -2a \end{cases}$$

thus, $f(x,y) = \frac{1}{3}x^3 + axy^2 - 2x^2 - 2axy + 6x + y^{2020}$

Since $\frac{2}{3} = \frac{1}{3} \cdot 2^3 + a \cdot 2 \cdot 1^2 - 2 \cdot 2^2 - 2 \cdot a \cdot 2 \cdot 1 + 6 \cdot 2 + 1$

$= \frac{8}{3} - 2a - 8 + 12 + 1 = \frac{8}{3} + 5 - 2a$, we

get $2a = \frac{8}{3} - \frac{2}{3} + 5 = 2 + 5 = 7 \Rightarrow \boxed{a = 7/2}$

$\boxed{c = -2a = -7}$