# Solutions to Math 105 Midterm 1 (Version I)
(Feb. 4, 2020)

(a) $\langle c, -1, -3\rangle \cdot \langle 3, 1, -1\rangle = 0 \Rightarrow 3c - 1 + 3 = 0 \Rightarrow c = -\frac{2}{3}.$

[3]

$$\boxed{\text{Answer: } -\tfrac{2}{3}}$$

(b) $2(x+2) + (y-1) - 6(z+3) = 0$.

[3]

Answer: $2(x+2)+(y-1)-6(z+3)=0$

or

$2x + y - 6z = 15$

(c) $f_y(x,y) = 2x^3y$. $f_{yx} = 6x^2y$, $f_{yx}(2,3) = 6\cdot 2^2 \cdot 3 = 72$

[3]

$$\boxed{\text{Answer: } 72}$$

(d) $f_y(2.45, 1.8) \approx \dfrac{f(2.45, 1.7) - f(2.45, 1.8)}{1.7 - 1.8} = \dfrac{2.3483 - 2.3494}{-0.1}$

[3]

$$\boxed{\text{Answer: } \dfrac{2.3483 - 2.3494}{-0.1}}$$

2. (a) [2] The domain of $f(x,y)$ is given by

$$\{(x,y):\ x^2+\frac{y^2}{4}\le 1\}.$$

(b) [3] Any level curve of $z=f(x,y)$ is given by

$f(x,y)=c$ for some constant $c$.

or $\sqrt{1-x^2-\frac{y^2}{4}}=c$.

- If $c<0$ or $c>1$, the level curve does not exist.
- If $c=1$, the level curve is the point $(0,0)$.
- If $0\le c<1$, the level curve is an ellipse.

(c) [4] $\frac{\partial z}{\partial y}=\frac{1}{2}\left(1-x^2-\frac{y^2}{4}\right)^{-1/2}\cdot\frac{-2y}{4}=-\frac{y}{4}\left(1-x^2-\frac{y^2}{4}\right)^{-1/2}$.

Hence, $\frac{\partial z}{\partial y}\left(\frac{1}{\sqrt{2}},1\right)=-\frac{1}{4}\left(1-\frac{1}{2}-\frac{1}{4}\right)^{-1/2}=-\frac{1}{4}\left(\frac{1}{4}\right)^{-1/2}\left(=-\frac{1}{2}\right)$.

3. (a) [3] $f_{xx} = (y^2-8y)\left[-e^{-x}+(1-x)e^{-x}(-1)\right] = (y^2-8y)e^{-x}(x-2)$.

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left((y^2-8y)(1-x)e^{-x}\right)$$

$$= (1-x)e^{-x}\cdot\frac{\partial}{\partial y}(y^2-8y) = (1-x)e^{-x}(2y-8)$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left((2y-8)xe^{-x}\right) = 2xe^{-x}$$

(b) [5] $D(x,y) = (y^2-8y)(2x)(x-2)e^{-2x} - (2y-8)^2(1-x)^2e^{-2x}$

$D(0,0) = -(0-8)^2 1^2 e^0 = -64 < 0$

$D(0,8) = -(16-8)^2 1^2 \cdot e^0 = -64 < 0$

$D(1,4) = (16-32)\cdot 2(-1)e^{-2} - 0 > 0$

$f_{xx}(1,4) = (16-32)e^{-1}(-1) > 0$

Hence, $f$ has a saddle point at $(0,0)$ and $(0,8)$,

$f$ has a local minimum at $(1,4)$.

4. (a) [3] $\frac{\partial}{\partial x}(x^3 - x + y^3 z + z^3) = \frac{\partial}{\partial x}(1)$

$$3x^2 - 1 + y^3 \cdot \frac{\partial z}{\partial x} + 3z^2 \cdot \frac{\partial z}{\partial x} = 0$$

$$(y^3 + 3z^2)\frac{\partial z}{\partial x} = 1 - 3x^2 \qquad (1)$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{1 - 3x^2}{y^3 + 3z^2}. \qquad (2)$$

(b) [2] If $(x, y) = (1, 0)$, then $z^3 = 1$. Hence, $z = 1$.

It follows from (2) that

$$\frac{\partial z}{\partial x}(1, 0) = \frac{1 - 3 \cdot 1^2}{0^3 + 3 \cdot 1^2} = -\frac{2}{3}.$$

(c) [4] $\frac{\partial}{\partial y}(x^3 - x + y^3 z + z^3) = \frac{\partial}{\partial y}(1)$

$$3y^2 \cdot z + y^3 \cdot \frac{\partial z}{\partial y} + 3z^2 \cdot \frac{\partial z}{\partial y} = 0$$

$$(y^3 + 3z^2)\frac{\partial z}{\partial y} = -3y^2 z \qquad (3)$$

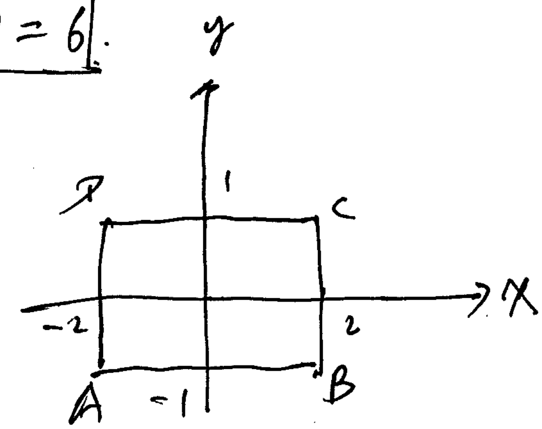
If $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$, then (1) and (3) give

$$\begin{cases} 1 - 3x^2 = 0 & (4) \\ -3y^2 z = 0 & (5). \end{cases}$$

Since $x^3 - x + y^3 z + z^3 = 1 \cdots (6)$, we get that $z \neq 0$ and $(\pm\frac{1}{\sqrt{3}}, 0)$ are two critical points

5. First, we find the values of $f$ at all critical points in the interior of $R$. We have
[8]

$f_x = -2x$, $f_y = -8y$, so $(0,0)$ is the only critical point in $R$ and $\boxed{f(0,0) = 6}$.




Next, find the minimum and maximum values of $f$ on the boundary of $R$.

On the sides $AB$ and $DC$, $y = \pm 1$, $-2 \le x \le 2$, we have $f(x, \pm 1) = 6 - x^2 - 4 = 2 - x^2$. $\frac{df}{dx} = -2x = 0$

$\Rightarrow x = 0$. $\boxed{f(0, \pm 1) = 2}$ $\boxed{f(\pm 2, \pm 1) = -2}$

On the sides $BC$ and $DA$, $x = \pm 2$, $-1 \le y \le 1$,

we have $f(\pm 2, y) = 6 - 4 - 4y^2 = 2 - 4y^2$.

$\frac{df}{dy} = -8y = 0 \Rightarrow y = 0$, $f(\pm 2, 0) = 2$, $f(\pm 2, \pm 1) = -2$.

Therefore, the maximum value of $f$ on $R$ is $6$, and the minimum value of $f$ on $R$ is $-2$.

6.[4] $f_x = x^2 + ay^2 + bx + cy + 6$

$f_{xx} = 2x + b$ we have $\begin{cases} f_{xx}(2,1) = 0 \\ f_{xy}(2,1) = 0 \end{cases}$ or

$f_{xy} = 2ay + c$

$\begin{cases} 2\cdot 2 + b = 0 \\ 2a + c = 0 \end{cases} \Rightarrow \begin{cases} \boxed{b = -4} \\ c = -2a \end{cases}$

thus, $f(x,y) = \frac{1}{3}x^3 + axy^2 - 2x^2 - 2axy + 6x + y^{2020}$

Since $\frac{2}{3} = \frac{1}{3}\cdot 2^3 + a\cdot 2\cdot 1^2 - 2\cdot 2^2 - 2\cdot a\cdot 2\cdot 1 + 6\cdot 2 + 1$

$= \frac{8}{3} - 2a - 8 + 12 + 1 = \frac{8}{3} + 5 - 2a$, we

get $2a = \frac{8}{3} - \frac{2}{3} + 5 = 2 + 5 = 7 \Rightarrow \boxed{a = 7/2}$

$\boxed{c = -2a = -7}$