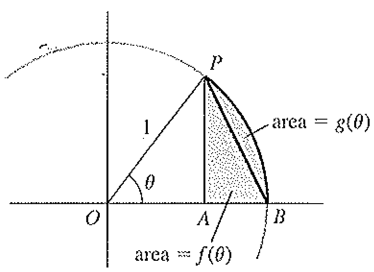
89. Logs with different bases Show that $f(x) = \log_a x$ and $g(x) = \log_b x$, where $a > 1$ and $b > 1$, grow at a comparable rate as $x \to \infty$.

90. Factorial growth rate The factorial function is usually defined for positive integers as $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. A valuable result that gives good approximations to $n!$ for large values of n is Stirling's formula, $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$ (see Guided Projects for more on Stirling's formula). Use this formula and a calculator to determine where the factorial function appears in the ranking of growth rates.

91. A geometric limit Let $f(\theta)$ be the area of the triangle ABP (see figure) and let $g(\theta)$ be the area of the region between the chord PB and the arc PB. Evaluate $\lim_{\theta \to 0} g(\theta)/f(\theta)$.



92. A fascinating function Consider the function $f(x) = (ab^x + (1-a)c^x)^{1/x}$, where a, b, and c are positive real numbers, $0 < a < 1$.

a. Graph f for several sets of (a, b, c). Verify that in all cases that f is an increasing function for all x with a single inflection point.

b. Use analytical methods to determine $\lim_{x \to 0} f(x)$ in terms of a, b, and c.

c. Use analytical methods to determine $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

d. Estimate the location of the inflection point (in terms of a, b, and c).

93. Exponential limit Prove that $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$ for $a \neq 0$.

94. Exponentials vs. super exponentials Show that x^x grows faster than b^x as $x \to \infty$ for $b > 1$.

95. Exponential growth rates

a. For what values of $b > 0$ does b^x grow faster than e^x as $x \to \infty$?

b. Compare the growth rates of e^x and e^{ax} as $x \to \infty$ for $a > 0$.

96. A max/min detector Consider the function $f(t) = (ax^t + by^t)^{1/t}$, where a, b, x, and y are positive real numbers with $a + b = 1$.

a. Show that $\lim_{t \to 0} f(t) = x^a y^b$.

b. Show that $\lim_{t \to \infty} f(t) = \max\{x, y\}$ and $\lim_{t \to -\infty} f(t) = \min\{x, y\}$.

QUICK CHECK ANSWERS

1. g and h **2.** g and h **3.** $0 \cdot \infty$; $(x - \pi/2)/\cot x$ **4.** The form 0^∞ (for example, $\lim_{x \to 0^+} x^{1/x}$) is not indeterminate, because as the base goes to zero, raising it to larger and larger powers drives the entire function to zero. **6.** x^3 grows faster than x^2 as $x \to \infty$, whereas x^2 and $10x^2$ have comparable growth rates as $x \to \infty$.

4.8 Antiderivatives

The goal of differentiation is to find the derivative f' of a given function f. The reverse process, called *antidifferentiation*, is equally important: Given a function f, we look for an *antiderivative* function F whose derivative is f; that is, a function F such that $F' = f$.

DEFINITION Antiderivative

A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$ for all x in I.

In this section, we revisit derivative formulas developed in previous chapters to discover corresponding antiderivative formulas.

Thinking Backward

Consider the function $f(x) = 1$ and the derivative formula $\frac{d}{dx}(x) = 1$. We see that an antiderivative of f is $F(x) = x$ because $F'(x) = 1 = f(x)$. Using the same logic, we can write

$$\frac{d}{dx}(x^2) = 2x \quad \Rightarrow \quad \text{an antiderivative of } f(x) = 2x \text{ is } F(x) = x^2$$

$$\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \text{an antiderivative of } f(x) = \cos x \text{ is } F(x) = \sin x.$$

QUICK CHECK 1 Verify by differentiation that x^3 is an antiderivative of $3x^2$ and $-\cos x$ is an antiderivative of $\sin x$. ◂

Each of these proposed antiderivative formulas is easily checked by showing that $F' = f$.

An immediate question arises: Does a function have more than one antiderivative? To answer this question, let's focus on $f(x) = 1$ and the antiderivative $F(x) = x$. Because the derivative of a constant C is zero, we see that $F(x) = x + C$ is also an antiderivative of $f(x) = 1$, which is easy to check:

$$F'(x) = \frac{d}{dx}(x + C) = 1 = f(x)$$

Therefore, $f(x) = 1$ actually has an infinite number of antiderivatives. For the same reason, any function of the form $F(x) = x^2 + C$ is an antiderivative of $f(x) = 2x$, and any function of the form $F(x) = \sin x + C$ is an antiderivative of $f(x) = \cos x$, where C is an arbitrary constant.

We might ask whether there are still *more* antiderivatives of a given function. The following theorem provides the answer.

THEOREM 4.16 The Family of Antiderivatives
Let F be any antiderivative of f. Then *all* the antiderivatives of f have the form $F + C$, where C is an arbitrary constant.

Proof Suppose that F and G are antiderivatives of f on an interval I. Then $F' = f$ and $G' = f$, which implies that $F' = G'$ on I. From Theorem 4.11, which states that functions with equal derivatives differ by a constant, it follows that $G = F + C$. Therefore, all antiderivatives of f have the form $F + C$, where C is an arbitrary constant. ◂

Theorem 4.16 says that while there are infinitely many antiderivatives of a function, they are all of one family, namely, those functions of the form $F + C$. Because the antiderivatives of a particular function differ by a constant, the antiderivatives are vertical translations of one another (Figure 4.78).

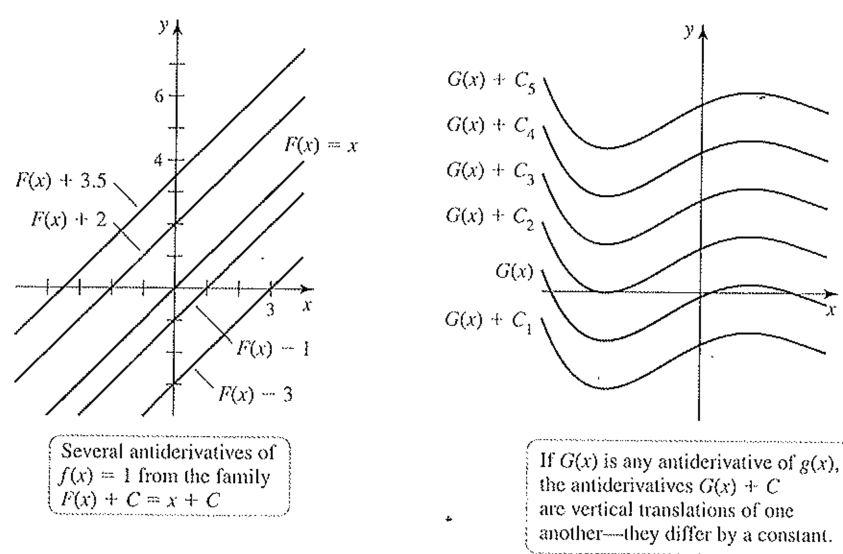


FIGURE 4.78

EXAMPLE 1 **Finding antiderivatives** Use what you know about derivatives to find all antiderivatives of the following functions.

a. $f(x) = 3x^2$ **b.** $f(x) = \dfrac{1}{1 + x^2}$ **c.** $f(x) = \sin x$

SOLUTION

a. Note that $\dfrac{d}{dx}(x^3) = 3x^2$. Reversing this derivative formula says that an antiderivative of $f(x) = 3x^2$ is x^3. By Theorem 4.16, the complete family of antiderivatives is $F(x) = x^3 + C$, where C is an arbitrary constant.

b. Because $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1 + x^2}$, all antiderivatives of f are of the form $F(x) = \tan^{-1} x + C$, where C is an arbitrary constant.

c. Recall that $\dfrac{d}{dx}(\cos x) = -\sin x$. We seek a function whose derivative is $\sin x$, not $-\sin x$. Observing that $\dfrac{d}{dx}(-\cos x) = \sin x$, it follows that the antiderivatives are $F(x) = -\cos x + C$, where C is an arbitrary constant. *Related Exercises 11–18* ◄

QUICK CHECK 2 Find the family of antiderivatives for each of $f(x) = e^x$, $g(x) = 4x^3$, and $h(x) = \sec^2 x$. ◄

Indefinite Integrals

The notation $\dfrac{d}{dx}(f)$ means *take the derivative of f*. We need analogous notation for antiderivatives. For historical reasons that become apparent in the next chapter, the notation that means *find the antiderivatives of f* is the **indefinite integral** $\int f(x)\,dx$. Every time an indefinite integral sign $\int$ appears, it is followed by a function called the **integrand**, which in turn is followed by the differential dx. For now dx simply means that x is the independent variable, or the **variable of integration**. The notation $\int f(x)\,dx$ represents *all* of the antiderivatives of f.

Using this new notation, the three results of Example 1 are written as

$$\int 3x^2\,dx = x^3 + C, \quad \int \frac{1}{1 + x^2}\,dx = \tan^{-1} x + C, \text{ and } \quad \int \sin x\,dx = -\cos x + C,$$

where C is an arbitrary constant called a **constant of integration**. Virtually all the derivative formulas presented earlier in the text may be written in terms of indefinite integrals. We begin with the Power Rule.

> **THEOREM 4.17** **Power Rule for Indefinite Integrals**
>
> $$\int x^p\,dx = \frac{x^{p+1}}{p + 1} + C,$$
>
> where $p \neq -1$ is a real number and C is an arbitrary constant.

> Notice that if $p = -1$ in this antiderivative formula, then $F(x)$ is undefined. The antiderivative of $f(x) = x^{-1}$ is discussed shortly.

Proof The theorem says that the antiderivatives of $f(x) = x^p$ are of the form $F(x) = \frac{x^{p+1}}{p+1} + C$. Differentiating F, we verify that $F'(x) = f(x)$:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{x^{p+1}}{p+1} + C\right) \\ &= \frac{d}{dx}\left(\frac{x^{p+1}}{p+1}\right) + \underbrace{\frac{d}{dx}(C)}_{0} \\ &= \frac{(p+1)x^{(p+1)-1}}{p+1} + 0 = x^p \end{aligned}$$

> Any indefinite integral calculation can be checked by differentiation: The derivative of the alleged indefinite integral must equal the integrand.

Theorems 3.4 and 3.5 (Section 3.2) state the Constant Multiple and Sum Rules for derivatives. Here are the corresponding antiderivative rules, which are proved by differentiation.

THEOREM 4.18 Constant Multiple and Sum Rules

Constant Multiple Rule: $$\int cf(x)\,dx = c\int f(x)\,dx$$

Sum Rule: $$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

EXAMPLE 2 Indefinite integrals Determine the following indefinite integrals.

a. $\int (3x^5 + 2 - 5x^{-3/2})\,dx$ **b.** $\int \left(\frac{4x^{19} - 5x^{-8}}{x^2}\right)dx$

SOLUTION

> $\int dx$ means $\int 1\,dx$, which is the indefinite integral of the constant function $f(x) = 1$, so $\int dx = x + C$.

> Each indefinite integral produces an arbitrary constant, all of which may be combined in one arbitrary constant called C.

a.
$$\begin{aligned} \int (3x^5 + 2 - 5x^{-3/2})\,dx &= \int 3x^5\,dx + \int 2\,dx - \int 5x^{-3/2}\,dx && \text{Sum Rule} \\ &= 3\int x^5\,dx + 2\int dx - 5\int x^{-3/2}\,dx && \text{Constant Multiple Rule} \\ &= 3\cdot\frac{x^6}{6} + 2\cdot x - 5\cdot\frac{x^{-1/2}}{(-1/2)} + C && \text{Power Rule} \\ &= \frac{x^6}{2} + 2x + 10x^{-1/2} + C && \text{Simplify.} \end{aligned}$$

b.
$$\begin{aligned} \int \left(\frac{4x^{19} - 5x^{-8}}{x^2}\right)dx &= \int (4x^{17} - 5x^{-10})\,dx && \text{Simplify the integrand.} \\ &= 4\int x^{17}\,dx - 5\int x^{-10}\,dx && \text{Sum and Constant Multiple Rules} \\ &= 4\cdot\frac{x^{18}}{18} - 5\cdot\frac{x^{-9}}{(-9)} + C && \text{Power Rule} \\ &= \frac{2x^{18}}{9} + \frac{5x^{-9}}{9} + C && \text{Simplify.} \end{aligned}$$

Both of these results should be checked by differentiation. *Related Exercises 19–26*

Indefinite Integrals of Trigonometric Functions

Any derivative formula can be restated in terms of an indefinite integral formula. For example, by the Chain Rule we know that

$$\frac{d}{dx}(\cos 3x) = -3 \sin 3x.$$

Therefore, we can immediately write

$$\int -3 \sin 3x\, dx = \cos 3x + C.$$

Factoring −3 from the left side and dividing through by −3, we have

$$\int \sin 3x\, dx = -\frac{1}{3} \cos 3x + C.$$

This argument works if we replace 3 by any constant $a \neq 0$. Similar reasoning leads to the results in Table 4.5, where $a \neq 0$ and C is an arbitrary constant.

Table 4.5 Indefinite Integrals of Trigonometric Functions

1.	$\frac{d}{dx}(\sin ax) = a \cos ax$	$\longrightarrow$	$\int \cos ax\, dx = \frac{1}{a} \sin ax + C$
2.	$\frac{d}{dx}(\cos ax) = -a \sin ax$	$\longrightarrow$	$\int \sin ax\, dx = -\frac{1}{a} \cos ax + C$
3.	$\frac{d}{dx}(\tan ax) = a \sec^2 ax$	$\longrightarrow$	$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax + C$
4.	$\frac{d}{dx}(\cot ax) = -a \csc^2 ax$	$\longrightarrow$	$\int \csc^2 ax\, dx = -\frac{1}{a} \cot ax + C$
5.	$\frac{d}{dx}(\sec ax) = a \sec ax \tan ax$	$\longrightarrow$	$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax + C$
6.	$\frac{d}{dx}(\csc ax) = -a \csc ax \cot ax$	$\longrightarrow$	$\int \csc ax \cot ax\, dx = -\frac{1}{a} \csc ax + C$

QUICK CHECK 3 Use differentiation to verify that $\int \sin 2x\, dx = -\frac{1}{2} \cos 2x + C.$ ◄

EXAMPLE 3 Indefinite integrals of trigonometric functions Determine the following indefinite integrals.

a. $\int \sec^2 3x\, dx$ **b.** $\int \cos\left(\frac{x}{2}\right) dx$

SOLUTION These integrals follow directly from Table 4.5 and can be verified by differentiation.

a. Letting $a = 3$ in result (3) of Table 4.5, we have

$$\int \sec^2 3x\, dx = \frac{\tan 3x}{3} + C.$$

b. We let $a = \frac{1}{2}$ in result (1) of Table 4.5, which says that

$$\int \cos\left(\frac{x}{2}\right) dx = \frac{\sin(x/2)}{\frac{1}{2}} + C = 2 \sin\left(\frac{x}{2}\right) + C.$$

Related Exercises 27–32 ◄

Other Indefinite Integrals

We now complete the process of rewriting familiar derivative results in terms of indefinite integrals. For example, because $\frac{d}{dx}(e^{ax}) = ae^{ax}$, where $a \neq 0$, we can divide both sides of this equation by a and write

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C.$$

Similarly, because $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x \neq 0$, it follows that $\int \frac{dx}{x} = \ln|x| + C$. Notice that this result fills the gap in the Power Rule for the case $p = -1$. The same reasoning leads to the indefinite integrals in Table 4.6, where $a \neq 0$ and C is an arbitrary constant.

➤ Tables 4.5 and 4.6 are subsets of the table of integrals at the end of the book.

Table 4.6 Other Definite Integrals

7. $\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\longrightarrow$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$
8. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$	$\longrightarrow$	$\int \frac{dx}{x} = \ln|x| + C$ (for $x \neq 0$)
9. $\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{a^2 - x^2}}$	$\longrightarrow$	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$ (for $|x| \leq |a|, a > 0$)
10. $\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{a}{x^2 + a^2}$	$\longrightarrow$	$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ (for all x and $a \neq 0$)
11. $\frac{d}{dx}\left(\sec^{-1}\left|\frac{x}{a}\right|\right) = \frac{a}{x\sqrt{x^2 - a^2}}$	$\longrightarrow$	$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$ (for $|x| \geq a > 0$)

EXAMPLE 4 Indefinite integrals Determine the following indefinite integrals.

a. $\int e^{-10x}\,dx$ **b.** $\int \frac{4}{\sqrt{9 - x^2}}\,dx$ **c.** $\int \frac{dx}{16x^2 + 1}$

SOLUTION

a. Setting $a = -10$ in result (7) of Table 4.6, we find that

$$\int e^{-10x}\,dx = -\frac{1}{10}e^{-10x} + C,$$

which should be verified by differentiation.

b. Setting $a = 3$ in result (9) of Table 4.6, we have

$$\int \frac{4}{\sqrt{9 - x^2}}\,dx = 4\int \frac{dx}{\sqrt{3^2 - x^2}} = 4\sin^{-1}\left(\frac{x}{3}\right) + C.$$

c. An algebra step is needed to put this integral in a form that matches Table 4.6. We first write

$$\int \frac{dx}{16x^2 + 1} = \frac{1}{16}\int \frac{dx}{x^2 + \left(\frac{1}{16}\right)} = \frac{1}{16}\int \frac{dx}{x^2 + \left(\frac{1}{4}\right)^2}.$$

Setting $a = \frac{1}{4}$ in result (10) of Table 4.6 gives

$$\int \frac{dx}{16x^2 + 1} = \frac{1}{16}\int \frac{dx}{x^2 + \left(\frac{1}{4}\right)^2} = \left(\frac{1}{16}\right) 4\tan^{-1} 4x + C = \frac{1}{4}\tan^{-1} 4x + C.$$

Related Exercises 33–38 ◀

Introduction to Differential Equations

Suppose you know that the derivative of a function f satisfies the equation

$$f'(x) = 2x + 10.$$

To solve this *differential equation* for the function f, we note that the solutions are antiderivatives of $2x + 10$, which are $x^2 + 10x + C$, where C is an arbitrary constant. So we have found an infinite number of solutions, all of the form $f(x) = x^2 + 10x + C$.

QUICK CHECK 4 Explain why an antiderivative of f' is f. ◂

Now consider a more general differential equation of the form $f'(x) = G(x)$, where G is given and f is unknown. The solution consists of antiderivatives of G, which involve an arbitrary constant. In most practical cases, the differential equation is accompanied by an **initial condition** that allows us to determine the arbitrary constant. Therefore, we consider problems of the form

$f'(x) = G(x)$,	where G is given	Differential equation
$f(a) = b$,	where a, b are given	Initial condition

A differential equation coupled with an initial condition is called an **initial value problem.**

EXAMPLE 5 An initial value problem Solve the initial value problem $f'(x) = x^2 - 2x$ with $f(1) = \frac{1}{3}$.

SOLUTION The solution is an antiderivative of $x^2 - 2x$. Therefore,

$$f(x) = \frac{x^3}{3} - x^2 + C,$$

where C is an arbitrary constant. We have determined that the solution is a member of a family of functions, all of which differ by a constant. This family of functions, called the **general solution**, is shown in Figure 4.79, where we see curves for various choices of C.

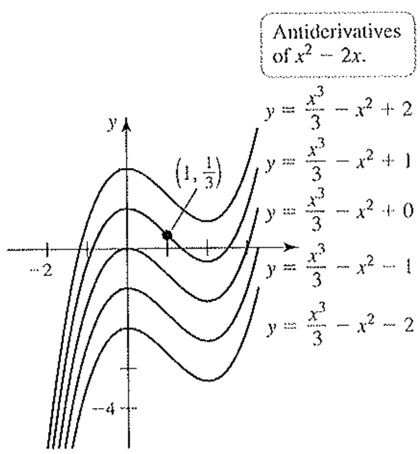


FIGURE 4.79

Using the initial condition $f(1) = \frac{1}{3}$, we must find the particular function in the general solution whose graph passes through the point $\left(1, \frac{1}{3}\right)$. Imposing the condition $f(1) = \frac{1}{3}$, we reason as follows:

$$
\begin{aligned}
f(x) &= \frac{x^3}{3} - x^2 + C && \text{General solution} \\
f(1) &= \frac{1}{3} - 1 + C && \text{Substitute } x = 1. \\
\frac{1}{3} &= \frac{1}{3} - 1 + C && f(1) = \tfrac{1}{3} \\
C &= 1 && \text{Solve for } C.
\end{aligned}
$$

Therefore, the solution to the initial value problem is

$$f(x) = \frac{x^3}{3} - x^2 + 1$$

which is just one of the curves in the family shown in Figure 4.79.

➤ It is advisable to check that the solution satisfies the original problem: $f'(x) = x^2 - 2x$ and $f(1) = \frac{1}{3} - 1 + 1 = \frac{1}{3}$.

Related Exercises 39–54 ◂

Motion Problems Revisited

QUICK CHECK 5 Position is an antiderivative of velocity. But there are infinitely many antiderivatives that differ by a constant. Explain how two objects can have the same velocity function but two different position functions. ◄

Antiderivatives allow us to revisit the topic of one-dimensional motion introduced in Section 3.5. Suppose the position of an object that moves along a line relative to an origin is $s(t)$, where $t \geq 0$ measures elapsed time. The velocity of the object is $v(t) = s'(t)$, which may now be read in terms of antiderivatives: *The position function is an antiderivative of the velocity.* If we are given the velocity function of an object and its position at a particular time, we can determine its position at all future times by solving an initial value problem.

➤ The convention with motion problems is to assume that motion begins at $t = 0$. This means that initial conditions are specified at $t = 0$.

We also know that the acceleration $a(t)$ of an object moving in one dimension is the rate of change of the velocity, which means $a(t) = v'(t)$. In antiderivative terms, this says that the velocity is an antiderivative of the acceleration. Thus, if we are given the acceleration of an object and its velocity at a particular time, we can determine its velocity at all times. These ideas lie at the heart of modeling the motion of objects.

Initial Value Problems for Velocity and Position

Suppose an object moves along a line with a (known) velocity $v(t)$ for $t \geq 0$. Then its position is found by solving the initial value problem

$$s'(t) = v(t), \ s(0) = s_0, \text{ where } s_0 \text{ is the initial position.}$$

If the acceleration of the object $a(t)$ is given, then its velocity is found by solving the initial value problem

$$v'(t) = a(t), \ v(0) = v_0, \text{ where } v_0 \text{ is the initial velocity.}$$

EXAMPLE 6 **A race** Runner A begins at the point $s(0) = 0$ and runs with velocity $v(t) = 2t$. Runner B begins with a head start at the point $S(0) = 8$ and runs with velocity $V(t) = 2$. Find the positions of the runners for $t \geq 0$ and determine who is ahead at $t = 6$ time units.

SOLUTION Let the position of Runner A be $s(t)$, with an initial position $s(0) = 0$. Then, the position function satisfies the initial value problem

$$s'(t) = 2t, \ s(0) = 0.$$

The solution is an antiderivative of $s'(t) = 2t$, which has the form $s(t) = t^2 + C$. Substituting $s(0) = 0$, we find that $C = 0$. Therefore, the position of Runner A is given by $s(t) = t^2$ for $t \geq 0$.

Let the position of Runner B be $S(t)$, with an initial position $S(0) = 8$. This position function satisfies the initial value problem

$$S'(t) = 2, \ S(0) = 8.$$

The antiderivatives of $S'(t) = 2$ are $S(t) = 2t + C$. Substituting $S(0) = 8$ implies that $C = 8$. Therefore, the position of Runner B is given by $S(t) = 2t + 8$ for $t \geq 0$.

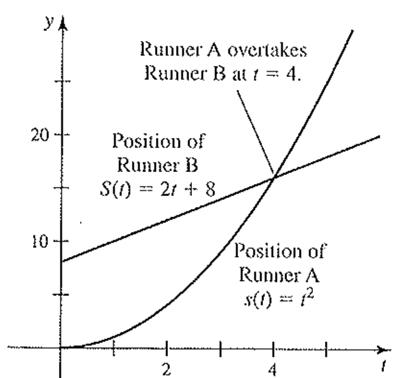


FIGURE 4.80

The graphs of the position functions are shown in Figure 4.80. Runner B begins with a head start but is overtaken when $s(t) = S(t)$, or when $t^2 = 2t + 8$. The solutions of this equation are $t = 4$ and $t = -2$. Only the positive solution is relevant because the race takes place for $t \geq 0$, so Runner A overtakes Runner B at $t = 4$, when $s = S = 16$. When $t = 6$, Runner A has the lead. *Related Exercises 55–62* ◄

EXAMPLE 7 **Motion with gravity** Neglecting air resistance, the motion of an object moving vertically near Earth's surface is determined by the acceleration due to gravity, which is approximately 9.8 m/s^2. Suppose a stone is thrown vertically upward at $t = 0$ with a velocity of 40 m/s from the edge of a cliff that is 100 m above a river.

a. Find the velocity $v(t)$ of the object for $t \geq 0$.

b. Find the position $s(t)$ of the object for $t \geq 0$.

c. Find the maximum height of the object above the river.

d. With what speed does the object strike the river?

SOLUTION We establish a coordinate system in which the positive s-axis points vertically upward with $s = 0$ corresponding to the river (Figure 4.81). Let $s(t)$ be the position of the stone measured relative to the river for $t \geq 0$. The initial velocity of the stone is $v(0) = 40$ m/s and the initial position of the stone is $s(0) = 100$ m.

➤ The acceleration due to gravity at Earth's surface is approximately $g = 9.8 \text{ m/s}^2$, or $g = 32 \text{ ft/s}^2$. It varies even at sea level from about 9.8640 at the poles to 9.7982 at the equator. The equation $v'(t) = -g$ is an instance of Newton's Second Law of Motion, assuming no other forces (such as air resistance) are present.

a. The acceleration due to gravity points in the *negative* s-direction. Therefore, the initial value problem governing the motion of the object is

$$\text{acceleration} = v'(t) = -9.8,\ v(0) = 40.$$

The antiderivatives of -9.8 are $v(t) = -9.8t + C$. The initial condition $v(0) = 40$ gives $C = 40$. Therefore, the velocity of the stone is

$$v(t) = -9.8t + 40.$$

As shown in Figure 4.82, the velocity decreases from its initial value $v(0) = 40$ until it reaches zero at the high point of the trajectory. This point is reached when

$$v(t) = -9.8t + 40 = 0$$

or when $t \approx 4.1$ s. For $t > 4.1$, the velocity becomes increasingly negative as the stone falls to Earth.

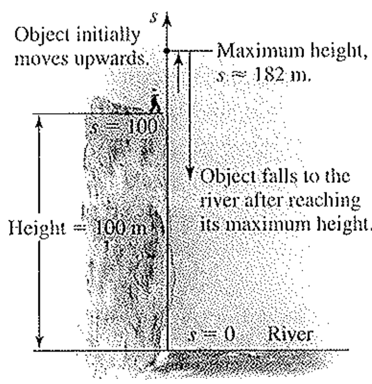


FIGURE 4.81

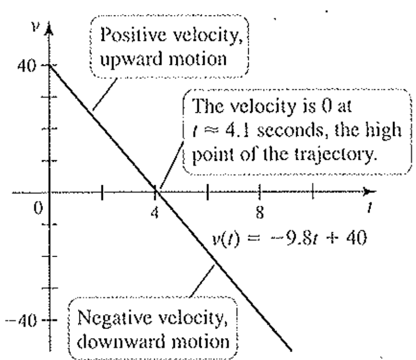


FIGURE 4.82

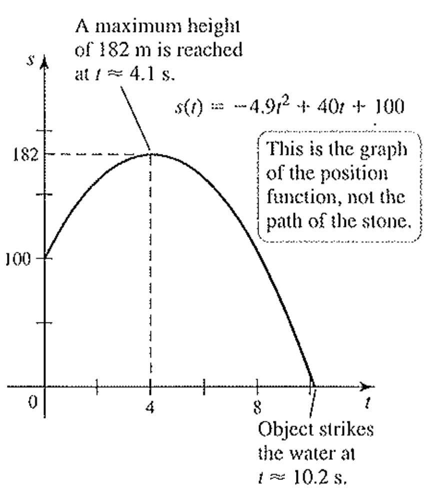


FIGURE 4.83

b. Knowing the velocity function of the stone, we can determine its position. The position function satisfies the initial value problem

$$v(t) = s'(t) = -9.8t + 40,\ s(0) = 100.$$

The antiderivatives of $-9.8t + 40$ are

$$s(t) = -4.9t^2 + 40t + C.$$

The initial condition $s(0) = 100$ implies $C = 100$, so the position function of the stone is

$$s(t) = -4.9t^2 + 40t + 100$$

as shown in Figure 4.83. The parabolic graph of the position function is not the actual trajectory of the stone; the stone moves vertically along the s-axis.

c. The position function of the stone increases for $0 < t < 4.1$. At $t \approx 4.1$, the stone reaches a high point of $s(4.1) \approx 182$ m.

d. For $t > 4.1$, the position function decreases, and the stone strikes the river when $s(t) = 0$. The roots of this equation are $t \approx 10.2$ and $t \approx -2.0$. Only the first root is relevant because the motion takes place for $t \geq 0$. Therefore, the stone strikes the ground at $t \approx 10.2$ s. Its speed at this instant is $|v(10.2)| \approx |-60| = 60$ m/s.

Related Exercises 63–66 ◄

SECTION 4.8 EXERCISES

Review Questions

1. Fill in the blank with the words *derivative* or *antiderivative*: If $F'(x) = f(x)$, then f is the ________ of F and F is the ________ of f.

2. Describe the set of antiderivatives of $f(x) = 0$.

3. Describe the set of antiderivatives of $f(x) = 1$.

4. Why do two different antiderivatives of a function differ by a constant?

5. Give the antiderivatives of x^p. For what values of p does your answer apply?

6. Give the antiderivatives of e^{-x}.

7. Give the antiderivatives of $1/x$ for $x > 0$.

8. Evaluate $\int \cos ax\,dx$ and $\int \sin ax\,dx$.

9. If $F(x) = x^2 - 3x + C$ and $F(-1) = 4$, what is the value of C?

10. For a given function f, explain the steps used to solve the initial value problem $F'(t) = f(t)$, $F(0) = 10$.

Basic Skills

11–18. Finding antiderivatives *Find all the antiderivatives of the following functions. Check your work by taking derivatives.*

11. $f(x) = 5x^4$

12. $g(x) = 11x^{10}$

13. $f(x) = \sin 2x$

14. $g(x) = -4\cos 4x$

15. $P(x) = 3\sec^2 x$

16. $Q(s) = \csc^2 s$

17. $f(y) = -2/y^3$

18. $H(z) = -6z^{-7}$

19–26. Indefinite integrals *Determine the following indefinite integrals. Check your work by differentiation.*

19. $\int (3x^5 - 5x^9)\,dx$

20. $\int (3u^{-2} - 4u^2 + 1)\,du$

21. $\int \left(4\sqrt{x} - \frac{4}{\sqrt{x}}\right) dx$

22. $\int \left(\frac{5}{t^2} + 4t^2\right) dt$

23. $\int (5s + 3)^2\,ds$

24. $\int 5m(12m^3 - 10m)\,dm$

25. $\int (3x^{1/3} + 4x^{-1/3} + 6)\,dx$

26. $\int 6\sqrt[3]{x}\,dx$

27–32. Indefinite integrals involving trigonometric functions *Determine the following indefinite integrals. Check your work by differentiation.*

27. $\int (\sin 2y + \cos 3y)\,dy$

28. $\int \left[\sin 4t - \sin\left(\frac{t}{4}\right)\right] dt$

29. $\int (\sec^2 x - 1)\,dx$

30. $\int 2\sec^2 2v\,dv$

31. $\int (\sec^2\theta + \sec\theta\tan\theta)\,d\theta$

32. $\int \frac{\sin\theta - 1}{\cos^2\theta}\,d\theta$

33–38. Other indefinite integrals *Determine the following indefinite integrals. Check your work by differentiation.*

33. $\int \frac{1}{2y}\,dy$

34. $\int (e^{2t} + 2\sqrt{t})\,dt$

35. $\int \frac{6}{\sqrt{25 - x^2}}\,dx$

36. $\int \frac{3}{4 + v^2}\,dv$

37. $\int \frac{1}{x\sqrt{x^2 - 100}}\,dx$

38. $\int \frac{2}{16z^2 + 25}\,dz$

39–42. Particular antiderivatives *For the following functions f, find the antiderivative F that satisfies the given condition.*

39. $f(x) = x^5 - 2x^{-2} + 1$; $F(1) = 0$

40. $f(t) = \sec^2 t$; $F(\pi/4) = 1$

41. $f(v) = \sec v \tan v$; $F(0) = 2$

42. $f(x) = (4\sqrt{x} + 6/\sqrt{x})/x^2$; $F(1) = 4$

43–48. Solving initial value problems *Find the solution of the following initial value problems.*

43. $f'(x) = 2x - 3$; $f(0) = 4$

44. $g'(x) = 7x^6 - 4x^3 + 12$; $g(1) = 24$

45. $g'(x) = 7x\left(x^6 - \frac{1}{7}\right)$; $g(1) = 2$

46. $h'(t) = 6\sin 3t$; $h(\pi/6) = 6$

47. $f'(u) = 4(\cos u - \sin 2u)$; $f(\pi/6) = 0$

48. $p'(t) = 10e^{-t}$; $p(0) = 100$

49–54. Graphing general solutions *Graph several functions that satisfy each of the following differential equations. Then, find and graph the particular function that satisfies the given initial condition.*

49. $f'(x) = 2x - 5,\ f(0) = 4$
50. $f'(x) = 3x^2 - 1,\ f(1) = 2$
51. $f'(x) = 3x + \sin \pi x,\ f(2) = 3$
52. $f'(s) = 4 \sec s \tan s,\ f(\pi/4) = 1$
53. $f'(t) = 1/t,\ f(1) = 4$
54. $f'(x) = 2 \cos 2x,\ f(0) = 1$

55–60. Velocity to position *Given the following velocity functions of an object moving along a line, find the position function with the given initial position. Then, graph both the velocity and position functions.*

55. $v(t) = 2t + 4;\ s(0) = 0$
56. $v(t) = e^{-2t} + 4;\ s(0) = 2$
57. $v(t) = 2\sqrt{t};\ s(0) = 1$
58. $v(t) = 2 \cos t;\ s(0) = 0$
59. $v(t) = 6t^2 + 4t - 10;\ s(0) = 0$
60. $v(t) = 2 \sin 2t;\ s(0) = 0$

61–62. Races *The velocity function and initial position of Runners A and B are given. Analyze the race that results by graphing the position functions of the runners and finding the time and positions (if any) at which they first pass each other.*

61. A: $v(t) = \sin t,\ s(0) = 0$; B: $V(t) = \cos t,\ S(0) = 0$
62. A: $v(t) = 2e^{-t},\ s(0) = 0$; B: $V(t) = 4e^{-4t},\ S(0) = 10$

63–66. Motion with gravity *Consider the following descriptions of the vertical motion of an object subject only to the acceleration due to gravity.*

a. *Find the velocity of the object for all relevant times*
b. *Find the position of the object for all relevant times*
c. *Find the time when the object reaches its highest point (What is the height?)*
d. *Find the time when the object strikes the ground*

63. A softball is popped up vertically (from the ground) with a velocity of 30 m/s.
64. A stone is thrown vertically upward with a velocity of 30 m/s from the edge of a cliff 200 m above a river.
65. A payload is released at an elevation of 400 m from a hot-air balloon that is rising at a rate of 10 m/s.
66. A payload is dropped at an elevation of 400 m from a hot-air balloon that is descending at a rate of 10 m/s.

Further Explorations

67. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
 a. $F(x) = x^3 - 4x + 100$ and $G(x) = x^3 - 4x - 100$ are antiderivatives of the same function.
 b. If $F'(x) = f(x)$, then f is an antiderivative of F.
 c. If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + C$.
 d. $f(x) = x^3 + 3$ and $g(x) = x^3 - 4$ are derivatives of the same function.
 e. If $F'(x) = G'(x)$, then $F(x) = G(x)$.

68–75. Miscellaneous indefinite integrals *Determine the following indefinite integrals. Check your work by differentiation.*

68. $\int \left(\sqrt[3]{x^2} + \sqrt{x^3}\right) dx$
69. $\int \frac{e^{2x} - e^{-2x}}{2}\,dx$
70. $\int (4 \cos 4w - 3 \sin 3w)\,dw$
71. $\int (\csc^2 \theta + 2\theta^2 - 3\theta)\,d\theta$
72. $\int (\csc^2 \theta + 1)\,d\theta$
73. $\int \frac{1 + \sqrt{x}}{x}\,dx$
74. $\int \frac{2 + x^2}{1 + x^2}\,dx$
75. $\int \sqrt{x}\,(2x^6 - 4\sqrt[3]{x})\,dx$

76–79. Functions from higher derivatives *Find the function F that satisfies the following differential equations and initial conditions.*

76. $F''(x) = 1, F'(0) = 3, F(0) = 4$
77. $F''(x) = \cos x, F'(0) = 3, F(\pi) = 4$
78. $F'''(x) = 4x, F''(0) = 0, F'(0) = 1, F(0) = 3$
79. $F'''(x) = 672x^5 + 24x, F''(0) = 0, F'(0) = 2, F(0) = 1$

Applications

80. **Mass on a spring** A mass oscillates up and down on the end of a spring. Find its position s relative to the equilibrium position if its acceleration is $a(t) = \sin(\pi t)$, and its initial velocity and position are $v(0) = 3$ and $s(0) = 0$, respectively.

81. **Flow rate** A large tank is filled with water when an outflow valve is opened at $t = 0$. Water flows out at a rate given by $Q'(t) = 0.1(100 - t^2)$ gal/min for $0 \le t \le 10$ min.
 a. Find the amount of water $Q(t)$ that has flowed out of the tank after t minutes, given the initial condition $Q(0) = 0$.
 b. Graph the flow function Q for $0 \le t \le 10$.
 c. How much water flows out of the tank in 10 min?

82. **General headstart problem** Suppose that object A is located at $s = 0$ at time $t = 0$ and starts moving along the s-axis with a velocity given by $v(t) = 2at$, where $a > 0$. Object B is located at $s = c > 0$ at $t = 0$ and starts moving along the s-axis with a constant velocity given by $V(t) = b > 0$. Show that A always overtakes B at time
$$t = \frac{b + \sqrt{b^2 + 4ac}}{2a}.$$

Additional Exercises

83. **Using identities** Use the identities $\sin^2 x = (1 - \cos 2x)/2$ and $\cos^2 x = (1 + \cos 2x)/2$ to find $\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$.

84–87. Verifying indefinite integrals *Verify the following indefinite integrals by differentiation. These integrals are derived in later chapters.*

84. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\,dx = 2 \sin \sqrt{x} + C$

85. $\int \frac{x}{\sqrt{x^2 + 1}}\,dx = \sqrt{x^2 + 1} + C$

86. $\int x^2 \cos x^3\, dx = \frac{1}{3} \sin x^3 + C$

87. $\int \frac{x}{(x^2-1)^2}\, dx = -\frac{1}{2(x^2-1)} + C$

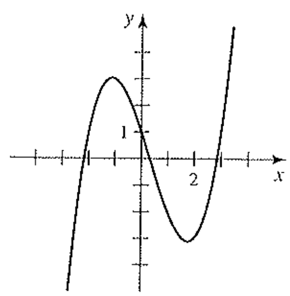
QUICK CHECK ANSWERS

1. $d/dx(x^3) = 3x^2$ and $d/dx(-\cos x) = \sin x$ 2. $e^x + C$, $x^4 + C$, $\tan x + C$ 3. $d/dx(-\cos(2x)/2 + C) = \sin 2x$ 4. One function that can be differentiated to get f' is f. Therefore, f is an antiderivative of f'. 5. The two position functions involve two different initial positions; they differ by a constant.

CHAPTER 4 REVIEW EXERCISES

1. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. If $f'(c) = 0$, then f has a local minimum or maximum at c.
   b. If $f''(c) = 0$, then f has an inflection point at $(c, f(c))$.
   c. $F(x) = x^2 + 10$ and $G(x) = x^2 - 100$ are antiderivatives of the same function.
   d. Between two local minima of a continuous function on $(-\infty, \infty)$, there must be a local maximum.

2. **Locating extrema** Consider the graph of a function f on the interval $[-3, 3]$.
   a. Identify the local minima and maxima of f.
   b. Identify the absolute minimum and absolute maximum of f (if they exist).
   c. Give the approximate coordinates of the inflection point(s) of f.
   d. Give the approximate coordinates of the zero(s) of f.
   e. On what intervals (approximately) is f concave up?
   f. On what intervals (approximately) is f concave down?

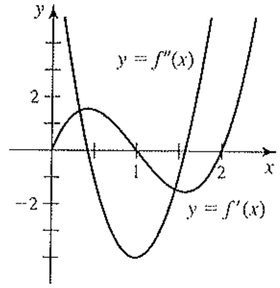


3–4. Designer functions *Sketch the graph of a continuous function that satisfies the following conditions.*

3. f is continuous on the interval $[-4, 4]$, $f'(x) = 0$ for $x = -2, 0,$ and 3; f has an absolute minimum at $x = 3$; f has a local minimum at $x = -2$; f has a local maximum at $x = 0$; f has an absolute maximum at $x = -4$.

4. f is continuous on $(-\infty, \infty)$; $f'(x) < 0$ and $f''(x) < 0$ on $(-\infty, 0)$; $f'(x) > 0$ and $f''(x) > 0$ on $(0, \infty)$.

5. **Functions from derivatives** Given the graphs of f' and f'', sketch a possible graph of f.



6–10. Critical points *Find the critical points of the following functions on the given intervals. Identify the absolute minimum and absolute maximum values (if possible). Graph the function to confirm your conclusions.*

6. $f(x) = \sin 2x + 3;\ [-\pi, \pi]$

7. $f(x) = 2x^3 - 3x^2 - 36x + 12;\ (-\infty, \infty)$

8. $f(x) = 4x^{1/2} - x^{5/2};\ [0, 4]$

9. $f(x) = 2x \ln x + 10;\ (0, 4)$

10. $g(x) = x^{1/3}(9 - x^2);\ [-4, 4]$

11. **Absolute values** Consider the function $f(x) = |x - 2| + |x + 3|$ on $[-4, 4]$. Graph f, identify the critical points, and give the coordinates of the local and absolute extreme values.

12. **Inflection points** Does $f(x) = 2x^5 - 10x^4 + 20x^3 + x + 1$ have any inflection points? If so, identify them.

13–20. Curve sketching *Use the guidelines of this chapter to make a complete graph of the following functions on their domains or on the given interval. Use a graphing utility to check your work.*

13. $f(x) = x^4/2 - 3x^2 + 4x + 1$

14. $f(x) = \frac{3x}{x^2 + 3}$

15. $f(x) = 4\cos[\pi(x - 1)]$ on $[0, 2]$

16. $f(x) = \frac{x^2 + x}{4 - x^2}$

17. $f(x) = \sqrt[3]{x} - \sqrt{x} + 2$

18. $f(x) = \frac{\cos \pi x}{1 + x^2}$ on $[-2, 2]$

19. $f(x) = x^{2/3} + (x + 2)^{1/3}$

20. $f(x) = x(x - 1)\, e^{-x}$

21. **Optimization** A right triangle has legs of length h and r and a hypotenuse of length 4 (see figure). It is revolved about the leg of length h to sweep out a right circular cone. What values of h and r maximize the volume of the cone?