The University of British Columbia

March 16, 2017

Common Midterm for All Sections of MATH 105 (Version 1)

Closed book examination   Time: 60 minutes

Last Name ______________________ **First** ______________

Signature ______________

Student Number ______________

MATH 105 Section Number: __________

Special Instructions:

No memory aids are allowed. No calculators. No communication devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

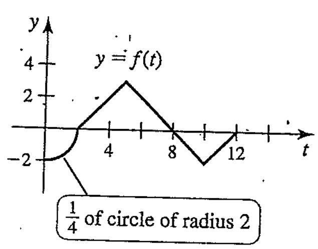
1		12
2		8
3		8
4		9
5		6
6		7
Total		50

[12] 1. **Short-Answer Questions: Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty.**

(a) [3] Fill in the blanks with left, right or midpoint, an interval, and a value of n in the statement below:

$\sum_{k=1}^{4} f(3+k)\cdot 1$ is a __________ Riemann sum for $f(x)$ on the interval [______, ______] with $n =$ __________.

(b) [3] Evaluate $\int_0^{12} f(t)\,dt$, where the graph of f is given in the following figure:



Answer:

(c) [3] Evaluate $\int \frac{x^2}{\sqrt{1-2x^3}}\,dx$.

Answer:

(d) [3] Evaluate $\int \frac{\ln x}{x^9}\,dx$.

Answer:

[8] **2**. Suppose that $\int_{-3}^{-2} f(x)\,dx = -2$, $\int_{-2}^{0} f(x)\,dx = 4$, $f(x) \leq 0$ on the interval $[-3,-2]$ and $f(x) \geq 0$ on the interval $[-2,0]$.

(a) Evaluate $\int_{-3}^{0} f(x)\,dx$.

(b) Evaluate $\int_{-3}^{0} |f(x)|\,dx$.

(c) Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} 3f\left(-2+(k-1)\frac{2}{n}\right)\frac{2}{n}$.

[8] **3.**

(a) Evaluate $\int \sin^3 x \cos^{2016} x \, dx$.

(b) Evaluate $\int \sec^2 x \, e^{\tan x} \, dx$.

(c) Approximate $\int_1^3 e^{\frac{1}{x}}\,dx$ using the Trapezoid rule with 4 subintervals.

(d) Evaluate $\int \frac{13x - 12}{x^2 - 3x + 2}\,dx$.

[9] 4. Evaluate the following integrals.

(a) $\int_8^\infty x^{-\frac{5}{3}}\, dx.$

(b) $\int_1^e (\ln x)^2\, dx.$

(c) $\int \sqrt{9 - x^2}\, dx.$

[6] **5.** Let $y = y(t)$ be a function defined on $\left[\frac{1}{2}, \infty\right)$. If $\frac{dy}{dt} = \frac{2t^2 + 4}{t} y$ and $y(1) = 1$, find the function $y = y(t)$.

[7] **6.**

(a) Suppose that $f(x)$ and $g(x)$ are continuous functions on $(-\infty, \infty)$ such that

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx \quad \text{for all real numbers } a \text{ and } b.$$

Is it possible that $f(x)$ and $g(x)$ are different functions? Please justify your answer.

(b) Let $F(x) = \displaystyle\int_0^x xe^{-t^2}\,dt$ for x in $(-\infty, \infty)$. Find the second derivative $F''(x)$ of $F(x)$.

$$1-\sin^2 x=\cos^2 x,\ 1+\tan^2 x=\sec^2 x,\ \sec^2 x-1=\tan^2 x$$

Half-angle formulas:

$$\cos^2 x=\frac{1+\cos 2x}{2},\ \sin^2 x=\frac{1-\cos 2x}{2}$$

Double-angle formulas:

$$\sin 2x = 2\sin x\cos x,$$

$$\cos 2x=\cos^2 x-\sin^2 x=2\cos^2 x-1=1-2\sin^2 x$$

Indefinite integrals:

$$\int \sec x\,dx=\ln|\sec x+\tan x|+C,\ \int \csc x\,dx=-\ln|\csc x+\cot x|+C$$

Summation identities:

$$\sum_{k=1}^{n} k=\frac{n(n+1)}{2},\ \sum_{k=1}^{n} k^2=\frac{n(n+1)(2n+1)}{6},\ \sum_{k=1}^{n} k^3=\frac{n^2(n+1)^2}{4}=\left(\frac{n(n+1)}{2}\right)^2.$$

Approximation's Rules and their error bound: Let $y=f(x)$ be a continuous function defined on the interval $[a,b]$ such that its derivatives f'' and $f^{(4)}$ are continuous. The midpoint Rule approximation $M(n)$, the Trapeziod Rule approximation $T(n)$ and the Simpson's Rule approximation $S(n)$ to $\int_a^b f(x)dx$ using n equally spaced subintervals on $[a,b]$ are given by the following formulas:

$$\begin{aligned} M(n) &= \sum_{k=1}^{n} f\left(a+(k-\frac{1}{2})\Delta x\right)\Delta x,\\ T(n) &= \left(\frac{1}{2}f(x_0)+\sum_{k=1}^{n-1} f(a+k\Delta x)+\frac{1}{2}f(x_n)\right)\Delta x,\\ S(n) &= (f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+4f(x_{n-1})+f(x_n))\frac{\Delta x}{3}, \end{aligned}$$

where $\Delta x=\frac{b-a}{n}$. The absolute errors in approximating the integral $\int_a^b f(x)dx$ by the Midpoint Rule, Trapezoid Rule satisfy the inequalities

$$E_M\le \frac{k(b-a)}{24}(\Delta x)^2 \text{ and } E_T\le\frac{k(b-a)}{12}(\Delta x)^2,$$

where k is a real number such that $|f''(x)|\le k$ for all x in $[a,b]$.

The absolute errors in approximating the integral $\int_a^b f(x)dx$ by the Simpson's Rule satisfies the inequalities

$$E_S \leq \frac{K(b-a)}{180}(\Delta x)^4$$

where K is a real number such that $|f^{(4)}(x)| \leq K$ for all x in $[a, b]$.

Equivalently, the error bounds can also be written as:

$$E_M \leq \frac{k(b-a)^3}{24n^2},$$
$$E_T \leq \frac{k(b-a)^3}{12n^2},$$
$$E_S \leq \frac{K(b-a)^5}{180n^4}.$$

(you can use any of these two versions for error bounds, as they give the same result.)

Solutions to Math 105 Midterm 2 (Version I)

1\. (a) [3] $\sum_{k=1}^{4} f(3+k)\cdot 1$ is a <u>right</u> (left) Riemann sum for $f(x)$ on the interval $[3, 7]$ ($[4, 8]$) with $n=$ <u>4</u>.

(b) [3] $-\frac{1}{4}\pi\cdot 2^2 + \frac{1}{2}\cdot 6\cdot 3 - \frac{1}{2}\cdot 4\cdot 2 = -\pi + 9 - 4 = 5-\pi$

(c) [3] $\int \frac{x^2}{\sqrt{1-2x^3}}\,dx \underset{\substack{u=1-2x^3\\ du=-6x^2dx}}{=} \int \frac{du/-6}{\sqrt{u}} = -\frac{1}{6}\int u^{-1/2}\,du$

$= -\frac{1}{6}\cdot\frac{u^{1/2}}{(1/2)} + C = -\frac{1}{3}\sqrt{1-2x^3} + C$

(d) [3] $\int \frac{\ln x}{x^9} \underset{\substack{u=\ln x,\ v'=x^{-9}\\ u'=\frac{1}{x},\ v=-\frac{1}{8}x^{-8}}}{=} -\frac{\ln x}{8x^8} - \int \frac{1}{x}\left(-\frac{1}{8}x^{-8}\right)dx$

$= -\frac{\ln x}{8x^8} + \frac{1}{8}\int x^{-9}\,dx = -\frac{\ln x}{8x^8} - \frac{1}{64}\frac{1}{x^8} + C$

2 (a) [2] $\int_{-3}^{0} f(x)dx = \int_{-3}^{-2} f(x)dx + \int_{-2}^{0} f(x)dx = -2+4 = 2$

(b) [3] $\int_{-3}^{0} |f(x)|dx = \int_{-3}^{-2} |f(x)|dx + \int_{-2}^{0} |f(x)|dx$

$= \int_{-3}^{-2} -f(x)dx + \int_{-3}^{0} f(x)dx = -\int_{-3}^{-2} f(x)dx + \int_{-3}^{0} f(x)dx$

$= -(-2)+4 = 6.$

(c) [3] Since $\sum_{k=1}^{n} 3f\left(-2+(k-1)\frac{2}{n}\right)\frac{2}{n}$ is the left Riemann sum for $3f(x)$ on $[-2,0]$, we have

$\lim_{n\to\infty} \sum_{k=1}^{n} 3f\left(-2+(k-1)\frac{2}{n}\right)\frac{2}{n} = \int_{-2}^{0} 3f(x)dx = 3\int_{-2}^{0} f(x)dx$

$= 3(4) = 12.$

3. (a) [2] $\int \sin^3 x \cos^{2016} x\,dx = \int \sin^2 x \sin x \cos^{2016} x\,dx$

$= \int (1-\cos^2 x)\cos^{2016} x \sin x\,dx \underset{\substack{u=\cos x \\ du=-\sin x\,dx}}{=} \int (1-u^2)u^{2016}(-du)$

$= -\int (u^{2016} - u^{2018})\,du = -\frac{u^{2017}}{2017} + \frac{u^{2019}}{2019} + C$

$= -\frac{\cos^{2017} x}{2017} + \frac{\cos^{2019} x}{2019} + C$

(b) [2] $\int \sec^2 x\, e^{\tan x}\,dx \underset{\substack{u=\tan x \\ du=\sec^2 x\,dx}}{=} \int e^u\,du = e^u + C$

$= e^{\tan x} + C$

(c) [2] $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$. Our grid points are $1, \frac{3}{2}, 2, \frac{5}{2}, 3$. Using the formula, we get

$$\int_1^3 e^{1/x}\,dx \approx \frac{1}{2}\left(\frac{1}{2}e^1 + e^{2/3} + e^{1/2} + e^{2/5} + \frac{1}{2}e^{1/3}\right).$$

(d) [2] $\frac{13x-12}{x^2-3x+2} = \frac{13x-12}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

$13x - 12 = A(x-2) + B(x-1)$

$x=2 \Rightarrow B=14$; $x=1 \Rightarrow A=-1$.

$\int \frac{13x-12}{x^2-3x+2}\,dx = \int \frac{-1}{x-1}\,dx + \int \frac{14}{x-2}\,dx = -\ln|x-1| + 14\ln|x-2| + C$.

4. (a) [3] $\int_8^\infty x^{-5/3}\,dx = \lim_{t\to\infty}\int_8^t x^{-5/3}\,dx$

$= \lim_{t\to\infty}\left(-\frac{3}{2}x^{-\frac{2}{3}}\right)\Big|_8^t = \lim_{t\to\infty}\left(-\frac{3}{2t^{2/3}} + \frac{3}{2\cdot 8^{2/3}}\right)$

$= -0 + \frac{3}{2\cdot 2^2} = \frac{3}{8}.$

(b) [3] $\int_1^e (\ln x)^2\,dx \underset{\substack{u=(\ln x)^2,\ v'=1\\ u'=2(\ln x)\frac{1}{x},\ v=x}}{=\!=\!=} (x(\ln x)^2]\Big|_1^e - 2\int_1^e \ln x\,dx$

$\underset{\substack{u=\ln x,\ v'=1\\ u'=\frac{1}{x},\ v=x}}{=\!=} e-2\left((x\ln x)\Big|_1^e - \int_1^e dx\right) = e-2\left(e - x\Big|_1^e\right)$

$= e-2\left(e-(e-1)\right) = e-2.$

(c) [3] $\int\sqrt{9-x^2}\,dx \underset{\substack{x=3\sin\theta\\ dx=3\cos\theta\,d\theta}}{=\!=\!=} \int\sqrt{9-(3\sin\theta)^2}\,3\cos\theta\,d\theta$

$= \int 3\cos\theta\,3\cos\theta\,d\theta = 9\int\cos^2\theta\,d\theta = 9\int\frac{1+\cos 2\theta}{2}\,d\theta$

$= \frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{9}{2}(\theta + \sin\theta\cos\theta) + C =$

$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} + \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C.$

5. [6] $\frac{dy}{y} = \frac{2t^2+4}{t}\,dt$

$\int \frac{dy}{y} = \int \left(2t + \frac{4}{t}\right) dt$

$\ln|y| = t^2 + 4\ln t + C$

$|y| = e^{t^2+4\ln t + C}$

$y = (\pm e^C)\, e^{t^2+4\ln t} = k\, e^{t^2+4\ln t}, \quad k = \pm e^C$

$1 = y(1) = k\, e^{1+4(0)} = ke \Rightarrow k = \frac{1}{e}$

Hence, $y = \frac{1}{e} e^{t^2+4\ln t} \left(= t^4 e^{t^2-1}\right)$.

6. (a) NO. In fact, for any t in $(-\infty, \infty)$, we have
[3]

$$\int_a^t f(x)\,dx = \int_a^t g(x)\,dx$$

by the assumption. Hence, we get

$$f(t) = \frac{d}{dt}\left(\int_a^t f(x)\,dx\right) = \frac{d}{dt}\left(\int_a^t g(x)\,dx\right) = g(t).$$

(b) $F'(x) = \left(x\int_0^x e^{-t^2}\,dt\right)' = \int_0^x e^{-t^2}\,dx + x e^{-x^2}$
[4]
$F''(x) = e^{-x^2} + e^{-x^2} + x e^{-x^2}(-2x) = (2 - 2x^2)e^{-x^2}.$