The University of British Columbia

Final Examination - April 23, 2015

Mathematics 105

All Sections

Closed book examination

Time: 2.5 hours

Last Name ____________________ **First** ______________ **Signature** ________________________

Student Number __________ **Section Number** __________ **Instructor** ____________________

Special Instructions:

No books, notes, or calculators are allowed. A formula sheet is included.

Senate Policy: Conduct during examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1a,b,c,d,e		15
1f,g,h,i,j		15
1k,l,m,n		12
2		12
3		12
4		10
5		16
6		8
Total		100

[42] **1. Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty.

(a) Find an equation of the plane parallel to the plane $x - 3y + 2z = 1$ passing through the point $(1, 0, -1)$.

Answer:

(b) Are the level curves of the paraboloid $z = x^2 + y^2$ lines, circles, parabolas, hyperbolas or ellipses?

Answer:

(c) Let $f(x, y) = y^3 \cos(2x)$. Find $\dfrac{\partial^2 f}{\partial x \partial y}$.

Answer:

(d) $\sum_{k=1}^{4} f(1+k) \cdot 1$ is a left Riemann sum for a function $f(x)$ on the interval $[a, b]$ with n sub intervals. Find the values of a, b and n.

Answer:
a= , b= , n=

(e) Suppose $\int_2^3 f(x)\,dx = -1$ and $\int_2^3 g(x)\,dx = 5$. Evaluate $\int_2^3 \big(6f(x) - 3g(x)\big)\,dx$.

Answer:

(f) For the function $f(x) = x^3 - \sin 2x$, find its antiderivative $F(x)$ that satisfies $F(0) = 1$.

Answer:

(g) Evaluate $\frac{d}{dx}\left(\int_0^{\sin x}(t^6+8)\,dt\right)$.

Answer:

(h) Evaluate $\int \frac{dx}{\sqrt{x^2+25}}$.

Answer:

(i) Evaluate $\int_0^{\frac{\pi}{2}} x\cos x\,dx$.

Answer:

(j) Evaluate $\int \cos^3 x \, dx$.

Answer:

(k) Evaluate the integral $\int_0^1 \frac{x^4}{x^5 - 1} \, dx$ or state that it diverges.

Answer:

(l) Evaluate $\sum_{k=7}^{\infty} \frac{1}{8^k}$.

Answer:

(m) Solve the differential equation $y'(t) = e^{\frac{y}{3}} \cos t$. You should express the solution $y(t)$ in terms of t explicitly.

Answer:

(n) Find the limit of the sequence $\left\{ \ln\left(\sin\frac{1}{n}\right) + \ln(2n) \right\}$.

Answer:

Full-Solution Problems. In questions 2 – 6, justify your answers and show all your work.

[12] **2**. (a) Evaluate $\displaystyle\int \frac{e^x}{(e^x+1)(e^x-3)}\,dx$.

2.(b) Evaluate $\displaystyle\int_2^4 \frac{x^2 - 4x + 4}{\sqrt{12 + 4x - x^2}}\,dx.$

[12] **3**. Let $f(x, y) = (x-1)^2 + (y+1)^2$.

(a) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x, y)$ on the circle $x^2 + y^2 = 4$. A solution that does not use the method of Lagrange multipliers will receive no credit, even if the answer is correct.

3.(b) Find the maximum and minimum values of the function $f(x, y)$ over the region $R = \{(x, y) : x^2 + y^2 \leq 4\}$.

[10] **4.** A continuous random variable X is given by the following probability density function

$$f(x) = \begin{cases} \frac{1}{4} + \frac{1}{2}|x| & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the expected value $E(X)$ of the random variable X.

(b) Let $F(x)$ be the cumulative distribution function for the random variable X. Find $F(x)$ for $0 < x < 1$.

[16] **5.**

(a) Suppose that $\dfrac{df}{dx} = \dfrac{x}{1+3x^3}$ and $f(0) = 1$. Find the Maclaurin series for $f(x)$.

(b) Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+n+1}{n^5-n}$ converges or diverges.

5.(c) Determine whether the series $\sum_{m=1}^{\infty} \frac{3m + \sin\sqrt{m}}{m^2}$ converges or diverges.

5.(d) Determine whether the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ converges or diverges.

[8] **6**. Suppose that the series $\sum_{n=0}^{\infty}(1-a_n)$ converges, where $a_n > 0$ for $n = 0, 1, 2, 3, \cdots$.

(a) Determine whether the series $\sum_{n=0}^{\infty} 2^n a_n$ converges or diverges.

(b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$.

Math 105 Final Exam. (April 23, 2015)

1. (a) $1\cdot(x-1)-3(y-0)+2(z+1)=0$ or $x-3y+2z=-1$.

(b) circles.

(c) $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(y^3\cos 2x)\right)=\frac{\partial}{\partial y}\left(y^3(-2)\sin 2x\right)=3y^2(-2)\sin 2x$

$= -6y^2\sin 2x.$

(d) $a=2$, $b=6$, $n=4$.

(e) $\int_2^3(6f-3g)\,dx = 6\int_2^3 f\,dx - 3\int_2^3 g(x)$

$=6(-1)-3\cdot 5=-21$

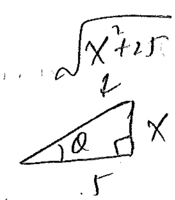
(f) $F(x)=\int f(x)\,dx=\int(x^3-\sin 2x)\,dx=\frac{x^4}{4}+\frac{1}{2}\cos 2x+c$

$1=F(0)=\frac{1}{2}\cos 0+c=\frac{1}{2}+c \Rightarrow c=\frac{1}{2}$. Hence,

$F(x)=\frac{x^4}{4}+\frac{1}{2}\cos 2x+\frac{1}{2}$.

(g) $(\sin^6 x+8)\cos x$.

(h) $\int\frac{dx}{\sqrt{x^2+25}} \underset{x=5\tan\theta}{=} \int\frac{5\sec^2\theta\,d\theta}{\sqrt{(5\tan\theta)^2+25}}=\int\sec\theta\,d\theta$



$=\ln|\sec\theta+\tan\theta|+C=\ln\left|\frac{\sqrt{x^2+25}}{5}+\frac{x}{5}\right|+C.$

(i) $\int_0^{\pi/2} x\cos x\,dx \underset{\substack{u=x,\ v'=\cos x\\ u'=1,\ v=\sin x}}{=} (x\sin x)\Big|_0^{\pi/2}-\int_0^{\pi/2}\sin x\,dx$

$=\frac{\pi}{2}+\cos x\Big|_0^{\pi/2}=\frac{\pi}{2}+\left(\cos\frac{\pi}{2}-\cos 0\right)=\frac{\pi}{2}-1.$

(j) $\int\cos^3 x\,dx=\int\cos x\cos^2 x\,dx=\int\cos x(1-\sin^2 x)\,dx \underset{\substack{u=\sin x\\ du=\cos x\,dx}}{=} \int(1-u^2)\,du$

$=u-\frac{u^3}{3}+C=\sin x-\frac{1}{3}\sin^3 x+C.$

1. (k) $\int_0^1 \frac{x^4}{x^5-1}dx = \lim_{t\to 1^-} \int_0^t \frac{x^4}{x^5-1}dx$

$= \lim_{t\to 1^-} \left(\frac{1}{5}\ln|x^5-1|\right)\Big|_0^t = \lim_{t\to 1^-} \frac{1}{5}\ln|t^5-1| = -\infty$,

so the integral is divergent

(l) $\left(\frac{1}{87}\right)/\left(1-\frac{1}{8}\right)$.

(m) $\frac{dy}{dt} = e^{y/3}\cos t \Rightarrow e^{-y/3}dy = \cos t\, dt$

$\int e^{-y/3}dy = \int \cos t\, dt \Rightarrow -3e^{-y/3} = \sin t + C$

$e^{-y/3} = \ln\left(-\frac{1}{3}\sin t - \frac{C}{3}\right) \Rightarrow y = -3\ln\left(-\frac{1}{3}\sin t - \frac{C}{3}\right)$.

(n) $\lim\left(\ln(\sin\frac{1}{n}) + \ln 2n\right) = \lim \ln\left(2n\sin\frac{1}{n}\right)$

$= \lim \ln\left(2\cdot\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) = \ln 2$.

2. (a) $\int \frac{e^x}{(e^x+1)(e^x-3)}\,dx \overset{u=e^x}{\underset{du=e^x dx}{=\!=}} \int \frac{du}{(u+1)(u-3)}$.

$\frac{1}{(u+1)(u-3)} = \frac{A}{u+1} + \frac{B}{u-3} \Rightarrow 1 = A(u-3)+B(u+1)$

$u=-1 \Rightarrow 1 = A(-4) \Rightarrow A = -1/4$.

$u=3 \Rightarrow 1 = B(4) \Rightarrow B = 1/4$.

Hence, $\int \frac{e^x}{(e^x+1)(e^x-3)}\,dx = \int \left(\frac{-1/4}{u+1} + \frac{1/4}{u-3}\right) du$

$= -\frac{1}{4}\ln|u+1| + \frac{1}{4}\ln|u-3| + C = -\frac{1}{4}\ln|e^x+1| + \frac{1}{4}\ln|e^x-3| + C$.

(b) $\int_2^4 \frac{x^2-4x+4}{\sqrt{12+4x-x^2}}\,dx = \int_2^4 \frac{x^2-4x+4}{\sqrt{12-(x^2-4x)}}\,dx = \int_2^4 \frac{x^2-4x+4}{\sqrt{12-(x^2-4x+4-4)}}\,dx$

$= \int_2^4 \frac{(x-2)^2}{\sqrt{16-(x-2)^2}}\,dx \overset{x-2=4\sin\theta}{\underset{dx=4\cos\theta}{=\!=}} \int_0^{\pi/6} \frac{(4\sin\theta)^2}{\sqrt{16-(4\sin\theta)^2}} \cdot 4\cos\theta\,d\theta$

$= 4^2 \int_0^{\pi/6} \sin^2\theta\,d\theta = 16\int_0^{\pi/6} \frac{1-\cos 2\theta}{2}\,d\theta = 8\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/6} = 8\left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)$.

3. (a) Let $g(x,y) = x^2+y^2-4$. Then we need to solve the system

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \quad \text{or} \quad \begin{cases} 2(x-1) = \lambda(2x) & (1) \\ 2(y+1) = \lambda(2y) & (2) \\ x^2+y^2 = 4 & (3) \end{cases}$$

$(1)\, y \Rightarrow \quad xy - y = \lambda xy \quad (4)$

$(2)\, x \Rightarrow \quad xy + x = \lambda xy \quad (5)$

(4) and $(5) \Rightarrow \quad y = -x. \quad (6)$

By (3) and (6), we get $x^2+(-x)^2 = 4 \Rightarrow x = \pm\sqrt{2} \Rightarrow y = \mp\sqrt{2}$.

$f(\sqrt{2}, -\sqrt{2}) = (\sqrt{2}-1)^2 + (-\sqrt{2}+1)^2 = 2(\sqrt{2}-1)^2 = 2(2-2\sqrt{2}+1) = 6-4\sqrt{2}$.

$f(-\sqrt{2}, \sqrt{2}) = (-\sqrt{2}-1)^2 + (\sqrt{2}+1)^2 = 2(\sqrt{2}+1)^2 = 2(2+2\sqrt{2}+1) = 6+4\sqrt{2}$.

Hence, the maximum value of $f(x,y)$ on the circle is $6+4\sqrt{2}$, the minimum value of $f(x,y)$ on the circle is $6-4\sqrt{2}$.

(b) $$\begin{cases} \dfrac{\partial f}{\partial x} = 2(x-1) = 0 \\ \dfrac{\partial f}{\partial y} = 2(y+1) = 0 \end{cases} \Rightarrow$$ $(1,-1)$ is the only critical point. $f(1,-1) = (1-1)^2 + (-1+1)^2 = 0. \quad (7)$

By the conclusions in (a) and (7), we know that the maximum value of $f(x,y)$ on the region R is $6+4\sqrt{2}$, the minimum value of $f(x,y)$ on the region R is 0.

//

5—(4)

4. (a) $E(X)=\int_{-\infty}^{\infty} x f(x)\,dx = \int_{-1}^{1} x\left(\frac{1}{4}+\frac{1}{2}|x|\right)dx$

$= \int_{-1}^{0} x\left(\frac{1}{4}-\frac{1}{2}x\right)dx + \int_{0}^{1} x\left(\frac{1}{4}+\frac{1}{2}x\right)dx$

$= \int_{-1}^{0}\left(\frac{1}{4}x-\frac{1}{2}x^2\right)dx + \int_{0}^{1}\left(\frac{1}{4}x+\frac{1}{2}x^2\right)dx$

$= \left(\frac{1}{4}\frac{x^2}{2}-\frac{1}{2}\cdot\frac{1}{3}x^3\right)\Big|_{-1}^{0} + \left(\frac{1}{4}\frac{x^2}{2}+\frac{1}{2}\cdot\frac{1}{3}x^3\right)\Big|_{0}^{1}$

$= -\left(\frac{1}{8}+\frac{1}{6}\right)+\left(\frac{1}{8}+\frac{1}{6}\right)=0.$

(b) $F(x)=\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} f(t)\,dt + \int_{-1}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$

$= 0+\int_{-1}^{0}\left(\frac{1}{4}-\frac{1}{2}t\right)dt + \int_{0}^{x}\left(\frac{1}{4}+\frac{1}{2}t\right)dt$

$= \left(\frac{1}{4}t-\frac{1}{2}\frac{t^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{1}{4}t+\frac{1}{2}\frac{t^2}{2}\right)\Big|_{0}^{x}$

$= -\left(-\frac{1}{4}-\frac{1}{4}\right)+\frac{1}{4}x+\frac{1}{4}x^2 = \frac{1}{2}+\frac{1}{4}x+\frac{1}{4}x^2.$

5. (a) $f'(x) = x \cdot \frac{1}{1-(-3x^3)} = x \sum_{n=0}^{\infty} (-3x^3)^n = \sum_{n=0}^{\infty} (-1)^n 3^n x^{3n+1}$

$f(x) = \int \left(\sum_{n=0}^{\infty} (-1)^n 3^n x^{3n+1} \right) dx = C + \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{3n+2}}{3n+2}$.

With $f(0) = 1$, we have $C = 1$ so $f(x) = 1 + \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{3n+2}}{3n+2}$.

(b) $\lim_{n\to\infty} \frac{(n^2+n+1)/(n^5-n)}{(1/n^3)} = \lim \frac{n^5+n^4+n^3}{n^5-n} =$

$= \lim \frac{1+(1/n)+(1/n^2)}{1-\left(\frac{1}{n^4}\right)} = 1$. Since $\Sigma \frac{1}{n^3}$ converges,

$\sum_{n=0}^{\infty} \frac{n^2+n+1}{n^5-n}$ converges.

(c) Since $\frac{3m+\sin\sqrt{m}}{m^2} > \frac{3m-1}{m^2}$ and

$\Sigma \frac{3m-1}{m^2}$ diverges, $\Sigma \frac{3m+\sin\sqrt{m}}{m^2}$ diverges

(or $\frac{3m+\sin\sqrt{m}}{m^2} > \frac{3m-m}{m^2} = \frac{2m}{m^2} = \frac{2}{m}$)

(d) $f(x) = \frac{1}{x(\ln x)^3}$ is positive, decreasing and continuous for $x \geq 2$. Since

$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^3}\,dx$

$\underset{\substack{u=\ln x \\ du=\frac{dx}{x}}}{=} \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^3} = \lim_{t\to\infty} \left(-\frac{1}{2u^2}\right)\Big|_{\ln 2}^{\ln t} = \frac{1}{2(\ln 2)^2}$,

the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ converges.

6. (a) Since $\sum_{n=0}^{\infty}(1-a_n)$ converges, $\lim(1-a_n)=0$ or $\lim a_n = 1$. Hence, $\lim(2^n a_n)=\infty$, which implies that $\sum 2^n a_n$ diverges.

(b) For $|x|>1$, $\lim|a_n x^n| = \lim(a_n |x|^n) = \infty$

Hence, $\lim a_n x^n \neq 0$. This proves that

$\sum a_n x^n$ diverges for $|x|>1$. (1).

Also, for $|x|<1$, with $x\neq 0$, we have

$$\lim\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim\left(\frac{a_{n+1}}{a_n}|x|\right) = \frac{\lim a_{n+1}}{\lim a_n}\cdot|x| = |x| < 1$$

$\Rightarrow$ $\sum a_n x^n$ converges for $|x|<1$ (2)

By (1) and (2), the radius of convergence is 1.